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E.E. BODY

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[Mathematical problems.

Vol. 1]

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$$\bar{x} = \frac{\int x\,dy}{\int dy}.$$

Now $xy^2 = a^3$

$$\therefore y = \frac{a^{\frac{3}{2}}}{x^{\frac{1}{2}}}.$$

$$\bar{x} = \frac{\int_0^a x \cdot \frac{a^{\frac{5}{2}}}{x^{\frac{1}{2}}}}{\int_0^a \frac{a^{\frac{3}{2}}}{x^{\frac{1}{2}}}} = \frac{\frac{2}{7}a^{\frac{7}{2}}}{\frac{2}{5}a^{\frac{5}{2}}} = \frac{10}{14}a = \frac{5}{7}a.$$

The whole of the curve is known at each end.

C & C' where x is the origin & B the axis of x.
& y the axis of y. If each the axis of x at
right angles at a distance a from the origin and
the axis of y is an asymptote to the curve.

$$\bar{x} = \frac{\int x\, dy}{\int x\, y}$$

$$\int x\, dy = b \int \frac{a-x}{x}.$$

$$\bar{x} = \frac{\int_0^a \sqrt{ax-x^2}}{\int_0^a \frac{\sqrt{a-x}}{x}\, dx}$$

$$\int \sqrt{ax-x^2} = \frac{x-\frac{a}{2}}{2}\sqrt{ax-x^2} + \frac{a^2}{8}\sin^{-1}\frac{2x}{a} + C$$

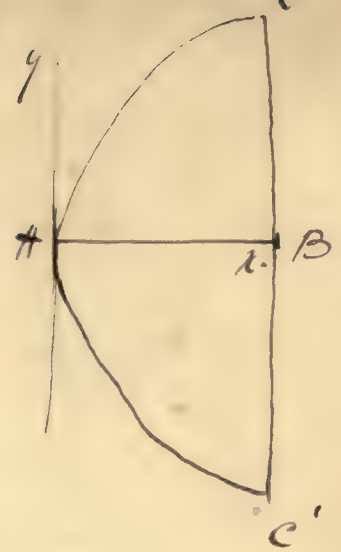
$$\therefore \int_0^a \sqrt{ax-x^2} = \frac{a^2}{8}\pi.$$

$$\text{also } \int \frac{\sqrt{a-x}}{\sqrt{x}} = \int \frac{\frac{a}{2}-x}{\sqrt{ax-x^2}} + \frac{a}{2}\int \frac{1}{\sqrt{ax-x^2}}$$

$$= \sqrt{ax-x^2} + \frac{a}{2}\sin^{-1}\frac{2x}{a} + C.$$

$$\therefore \int_0^a \frac{\sqrt{a-x}}{\sqrt{x}} = \frac{a\pi}{2}.$$

$$\text{therefore } \bar{x} = \frac{\frac{a^2\pi}{8}}{\frac{a\pi}{2}} = \frac{a}{4}.$$



Area $y = a \cos^{-1}\frac{x}{a} + \sqrt{2ax - x^2}$.

$$\therefore \bar{x} = \frac{\int_0^a x\,y}{\int_0^a y} = \frac{\int_0^a x\,dx\cos^{-1}\frac{x}{a} + x\sqrt{2ax-x^2}}{\int_0^a a\cos^{-1}\frac{x}{a} + \sqrt{2ax-x^2}}$$

$$\int x\cos^{-1}\frac{x}{a} = \frac{x^2}{2}\cos^{-1}\frac{x}{a} - \frac{a}{2}\int \frac{x^2}{\sqrt{2ax-x^2}}.$$

$$\frac{x^2}{\sqrt{2ax-x^2}} = -\int \frac{2ax-x^2}{\sqrt{2ax-x^2}} + \int \frac{(2x-2)}{\sqrt{2ax-x^2}} + \int \frac{2a^2}{\sqrt{2ax-x^2}}$$

$$= -\int \sqrt{2ax-x^2} - 2a\sqrt{2ax-x^2} + 2a^2\cos^{-1}\frac{x}{a}.$$

$$= -\left(\frac{x-a}{2}\right)\sqrt{2ax-x^2} + \frac{3a^2}{2}\cos^{-1}\frac{x}{a} - 2a\sqrt{2ax-x^2} + C.$$

$$-\frac{a}{2}\int \frac{x^2}{\sqrt{2ax-x^2}} = \frac{a}{2}\left(\frac{x+3a}{2}\right)\sqrt{2ax-x^2} - \frac{3a^3}{4}\cos^{-1}\frac{x}{a}.$$

$$\int x\sqrt{2ax-x^2} = \int -(a-x)\sqrt{2ax-x^2} + a\int\sqrt{2ax-x^2}$$

$$= -\frac{1}{3}(2ax-x^2)^{\frac{3}{2}} + \frac{a(x-a)}{2}\sqrt{2ax-x^2} + \frac{a^3}{2}\cos^{-1}\frac{x}{a}.$$

$\therefore$ Adding the results obtained.

$$\int\left(a\cos^{-1}\frac{x}{a} + x\sqrt{2ax-x^2}\right) = \frac{a}{2}\left(x^2 - \frac{a^2}{2}\right)\cos^{-1}\frac{x}{a} + \frac{1}{4}\sqrt{2ax-x^2}$$

$$\therefore \int_0^{2a} a\cos^{-1}\frac{a}{a} + x\sqrt{2ax-x^2} = \frac{a^3\pi}{4}.$$

Similarly $\int a\cos^{-1}\frac{x}{a} + \sqrt{2ax-x^2} = ax\cos^{-1}\frac{x}{a} + a\int\frac{a-x}{\sqrt{2ax-x^2}} - \frac{a^2}{2}\int\frac{1}{\sqrt{2ax-x^2}} + \int\sqrt{2a}$

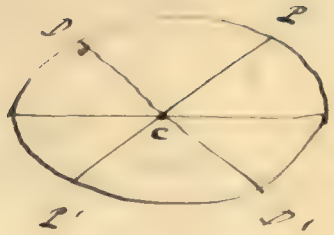
$$= \left(ax - \frac{a^2}{2}\right)\cos^{-1}\frac{x}{a} + a\sqrt{2ax-x^2} - \frac{a^2(x-a)}{2}\sqrt{2ax-x^2}$$

$$\therefore \int_0^{2a} a\cos^{-1}\frac{x}{a} + \sqrt{2ax-x^2} = \frac{3\pi a^2}{2}$$

$$\therefore \bar{x} = \frac{\frac{a^3\pi}{4}}{\frac{3\pi a^2}{2}} = \frac{14a}{12} = \frac{7}{6}a.$$

Let PP' be conjugate diameters

of an ellipse



$\therefore$ $y_2 = x$ locus of Q is $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

$\therefore$ locus of point of intersection of PD' & $P'D$ is

$$\bar{x} = \frac{\int_0^a x\, y}{\int_0^a y} = \frac{\int_0^a x \frac{b}{a}\sqrt{a^2-x^2}}{\int_0^a \frac{b}{a}\sqrt{a^2-x^2}}$$

$$= \frac{\int_0^a x\sqrt{a^2-x^2}}{\int_0^a \sqrt{a^2-x^2}} = \frac{4a}{3\pi}$$

$$\int x^2\sqrt{a^2-x^2} = -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}} + C \quad \therefore \int_0^a = -\frac{1}{3}a^3$$

$$\int \sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C \quad \therefore \int_0^a = \frac{a^2\pi}{4}$$

$$\bar{x} = \frac{\int_0^{a\sqrt{2}} \int_0^{\pi/4} r^2 \cos\theta \, dr \, d\theta}{\int_0^{a\sqrt{2}} \int_0^{\pi/4} r \, dr \, d\theta}$$

$$\int_r^a r^2 \cos\theta = \frac{1}{3} r^3 \cos\theta = \frac{a^3}{3} \cdot \left(1 - 2\sin^2\theta\right)^{\frac{3}{2}} \cos\theta.$$

$$\int_0^{\pi/4} \left(1 - 2\sin^2\theta\right)^{\frac{3}{2}} \cos\theta = \int_0^{1/\sqrt{2}} \left(1 - 2x^2\right)^{\frac{3}{2}} \quad \text{for } \cos\theta = dx.$$

$$\int_a \left(1 - 2x^2\right)^{\frac{3}{2}} = \frac{\sqrt{2}\, x \left(1 - 2x^2\right)^{\frac{3}{2}}}{4} + \frac{3}{4} \int x \left(1 - 2x^2\right)^{\frac{1}{2}}$$

$$= \frac{\sqrt{2}\, x \left(1 - 2x^2\right)^{\frac{3}{2}}}{4} + \frac{3\sqrt{2}\, x}{4} \cdot \left(1 - 2x^2\right)^{\frac{1}{2}} + \frac{3}{8\sqrt{2}} \sin^{-1} x\sqrt{2}.$$

$$\therefore \int_0^{1/\sqrt{2}} \left(1 - 2x^2\right)^{\frac{3}{2}} = \frac{3\sqrt{2}\,\pi}{8 \cdot 4}.$$

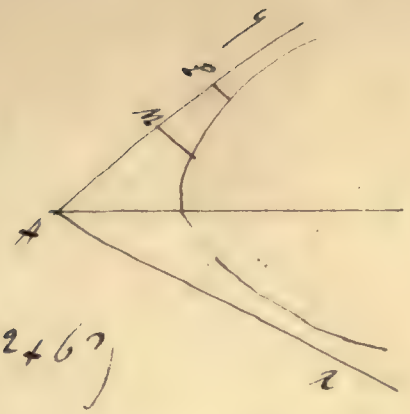
$$\text{Also } \int_\theta^{a\sqrt{2}\sec\theta} r = \frac{a^2}{2} \int_0^{\pi/4} \cos 2\theta = \frac{a^2}{4}.$$

$$\therefore \bar{x} = \frac{\frac{3\sqrt{2}\,\pi}{8 \cdot 4} \cdot \frac{a^3}{3}}{\frac{a^2}{4}} = \frac{a\sqrt{2}\,\pi}{8}.$$

The same result would be obtained by taking limits from $+\frac{\pi}{4}$ to $-\frac{\pi}{4}$.

Take A_y. for area of y.

$\frac{1}{2}r^2 = n$ cotes π is $A_y = \frac{1}{4}(a^2+b^2)$



$$\bar{x} = \frac{\int_{n}^{a} \int_{n}^{a'} x \, dy}{\int_{n}^{a} \int_{n}^{a'} dy} = \frac{\int_{n}^{a} \frac{1}{4}(a^2+b^2)}{\int_{n}^{a} \frac{1}{n} \frac{1}{4}(a^2+b^2)} = \frac{\frac{1}{4}(a^2+b^2)\{a-a'\}}{\frac{1}{4}(a^2+b^2)\{\log a - \log a'\}}$$

$$\therefore \bar{x} = \frac{a-a'}{\log a - \log a'}$$

$$\bar{y} = \frac{\int_{n}^{a} \int_{n}^{a'} y \, dy}{\int_{n}^{a} \int_{n}^{a'} dy} = \frac{\frac{1}{2}\int_{n}^{a} y^2}{\int_{n}^{a} y} = \frac{1}{\beta}\cdot(a^2+b^2)\cdot\frac{\int_{n}^{a} \frac{1}{x^2}}{\log a - \log a'}$$

$$= \frac{1}{\beta a a'}(a^2+b^2)\left\{\frac{a-a'}{\log a - \log a'}\right\}$$

Se n sia intero allora $y = \frac{x^m}{a^{m-1}}$.

$$\therefore \bar{x} = \frac{\int_0^a \int_y^{a'} x}{\int_0^a \int_y^{a'} 1} = \frac{\int_0^a x \cdot x^{m+1}}{\int_0^a x^m} = \frac{m+1}{m+2} \frac{a^{m+2} - a'^{m+2}}{a^{m+1} - a'^{m+1}}$$

$$\bar{y} = \frac{\int_0^a \int_y^{a'} y}{\int_0^a \int_y^{a'} 1} = \frac{\frac{1}{2}\int_0^a y^2}{\int_0^a y} = \frac{1}{2a^{m-1}} \cdot \frac{\int_0^a x \cdot x^{2m}}{\int_0^a x^m}$$

$$= \frac{1}{2a^{m-1}} \cdot \frac{m+1}{2m+1} \left\{ \frac{a^{2m+1} - a'^{2m+1}}{a^{m+1} - a'^{m+1}} \right\}.$$

The arc of the curve $y^2 = a^2 - x^2$

$$\bar{x} = \frac{\int_0^a \int_0^y x}{\int_0^a \int_0^y 1} = \frac{\int_0^a x\sqrt{a^2-x^2}}{\int_0^a \sqrt{a^2-x^2}} = \frac{4a}{3\pi}$$

for $\int x\sqrt{a^2-x^2} = \frac{1}{3}(a^2-x^2)^{\frac{3}{2}} + C$

$$\therefore \int_0^a x\sqrt{a^2-x^2} = \frac{1}{3}a^3.$$

also $\int \sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

$$\therefore \int_0^a = \frac{a^2\pi}{4} \quad \text{and} \quad \bar{x} = \frac{4a}{3\pi}.$$

$$\bar{y} = \frac{\int_0^a \int_0^y y}{\int_0^a \int_0^y 1} = \frac{1}{2}\frac{\int_0^a a^2-x^2}{\int_0^a \sqrt{a^2-x^2}} = \frac{\frac{1}{3}a^3}{\frac{a^2\pi}{4}} \text{ as before}$$

$$= \frac{4a}{3\pi}.$$

It might have been seen from the symmetrical form of the curve & must be equal to the ans. already.

$= n$ [illegible] curve $y = \sin x$.

$$\therefore \bar{x} = \frac{\int_0^{\pi} \int_y^{y} x}{\int_0^{\pi} y} = \frac{\int_0^{\pi} x \sin x}{\int_0^{\pi} \sin x}$$

$$\int x \sin x = \frac{x^2}{2} \sin x - \frac{1}{2} \int x^2 \cos x.$$

$$\int x \sin x = -x \cos x + \int \cos x.$$

$$= -x \cos x + \sin x.$$

$$\therefore \int_0^{\pi} x \sin x = \pi \quad \text{also} \int_0^{\pi} \sin x = 2$$

$$\therefore \bar{x} = \frac{\pi}{2}$$

$$\bar{y} = \frac{1}{2} \frac{\int_0^{\pi} y^2}{2} = \frac{1}{4} \int_0^{\pi} \sin^2 x.$$

$$\int \sin^2 x = -\frac{\sin x \cos x}{2} + \frac{x}{2} + C.$$

$$\therefore \int_0^{\pi} \sin^2 x = \frac{\pi}{2}$$

$$\therefore \bar{y} = \frac{\pi}{8}.$$

$$x^- = \frac{\int_x^{x'} x(y-y')}{\int_x^{x'} (y-y')} = \frac{\int_x^{x'} 2\sqrt{m}\, x^{\frac{3}{2}} - \beta x^2}{\int_x^{x'} 2\sqrt{m}\, x^{\frac{1}{2}} - \beta x}$$

$$= \frac{\frac{4}{5}\sqrt{m}\cdot x'^{\frac{5}{2}} - \frac{1}{3}\beta x'^3}{\frac{4}{3}\sqrt{m}\, x'^{\frac{3}{2}} - \frac{1}{2}\beta x'^2} = \frac{\frac{4}{5}\sqrt{m}\cdot x' - \frac{1}{3}\beta x'^{\frac{3}{2}}}{\frac{4}{3}\sqrt{m} - \frac{1}{2}\beta x'^{\frac{1}{2}}}$$

where x' = ordinate of the pt. of intersect. of the line with the $(-)$ $= \frac{4m}{\beta^2}$.

Similarly $y^- = \frac{1}{2}\frac{\int_x^{x'} (y^2 - y'^2)}{\int_x^{x'} (y-y')} = \frac{1}{2}\frac{\int_x^{x'} \{4mx - \beta^2 x^2\}}{\int_x^{x'} \left(2\sqrt{m}\cdot x^{\frac{1}{2}} - \beta x\right)}$

$$= \frac{1}{2}\cdot\left\{\frac{2mx'^2 - \frac{1}{3}\beta^2 x'^3}{\frac{4}{3}\sqrt{m}\cdot x'^{\frac{3}{2}} - \frac{1}{2}\beta x^2}\right\} = \frac{1}{2}\cdot\left\{\frac{2mx'^{\frac{1}{2}} - \frac{1}{3}\beta^2 x'^{\frac{3}{2}}}{\frac{4}{3}\sqrt{m} - \frac{1}{2}\beta x^{\frac{1}{2}}}\right\}$$

$= \frac{2m}{\beta}$ by substituting

$\therefore$ y alone makes of x'

ϕ = angle at the vertex of the sector

$$\therefore \bar{x} = \frac{\int_0^{\phi}\int_0^{a} r^3 \cos\theta \, dr \, d\theta}{\int_0^{\phi}\int_0^{a} r^2 \, dr \, d\theta} = \frac{3}{4} \cdot \frac{\int_0^{\phi} a^4 \cos\theta \, d\theta}{\int_0^{\phi} a^3 \, d\theta}$$

$$= \frac{3a}{8} \cdot \left(\frac{1-\cos^2\theta}{1-\cos\theta} \right) = \frac{3a}{8}(1+\cos\theta)$$

If $\theta = \frac{\pi}{2}$ the solid generated is a hemisphere

and $\cos\theta = 0$ $\therefore$ is a hemisphere $\bar{x} = \frac{3a}{8}$

If $\theta = \frac{\pi}{4}$ $\cos\theta = \frac{1}{\sqrt{2}}$

$$\therefore \bar{x} = \frac{3a}{8} \cdot \left(1 + \frac{1}{\sqrt{2}}\right).$$

$$y^{m+n} = x^m x^n \quad \therefore y^2 = x^{\frac{2m}{m+n}} \cdot x^{\frac{2n}{m+n}}$$

$$\bar{x} = \frac{\int_a^b x\, dy^2}{\int_a^b y^2} = \frac{\int_a^b x^{\frac{m+3n}{m+n}}}{\int_a^b x^{\frac{2n}{m+n}}} = \frac{\frac{m+n}{2m+4n} \cdot b^{\frac{2m+4n}{m+n}}}{\frac{m+n}{m+3n} \cdot b^{\frac{m+3n}{m+n}}}$$

$$= \frac{m+3n}{2m+4n} \cdot b.$$

If the curve pass through the origin $y^q = 4mx$.

$$\therefore m = n = 1$$

in which case the centroid $\bar{x} = \frac{2}{3} b$

If it is a parabola &c.

$$\int_{a'}^{a} x\, dy^2 = \frac{1}{3}\left\{ \frac{a^{13} - a'^3}{a^2 - a'^2} \right\} = \frac{2(a-a')}{3}\left\{ \frac{a^2 + aa' + a'^2}{a^2 - a'^2} \right\}$$

But $a' = \frac{b'^2}{4m}$ $\therefore \bar{x} = \frac{2h}{3}\left\{ \frac{b'^4 + b^2 b'^2 + b^4}{b'^4 - b^4} \right.$

$$= \frac{2h}{3}\left\{ b'^4 + 2b'^4 + b^4 \right.$$

$$\overline{X+x} = \frac{2}{3} \cdot \frac{x'^3 - x^3}{x'^2 - x^2}$$

$$= \frac{1}{3} \left\{ \frac{2x'^3 - 2x^3}{x'^2 - x^2} \right\}$$

$$\overline{X} = \frac{1}{3} \left\{ \frac{2x'^3 - 3x'^2 x + x^3}{x'^2 - x^2} \right\}$$

$$= \frac{1}{3} \frac{(x'^2 - 2x'x + x^2)(2x' + x)}{(x' - x)(x' + x)}$$

$$= \frac{1}{3} (x' - x) \left\{ \frac{2x' + x}{x' + x} \right.$$

$$= \frac{1}{3} h \left\{ \frac{8ax' + 6x'x}{4ax' + 6x^2} \right\}$$

$$= \frac{1}{3} x' \left\{ \frac{2b^2 + x^2}{b^2 + x^2} \right\}$$

$$y^2 = \frac{b^2}{a}(x^2 + 2ax).$$ for the limits $x=0$ $x=b$ values

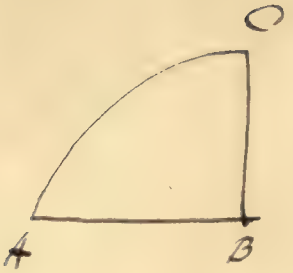
$$\therefore \bar{x} = \frac{\int_0^b x y^2}{\int_0^b y^2} = \frac{\int_0^b x^3 + 2ax^2}{\int_0^b x^2 + 2ax} = \frac{\frac{x^4}{4} + \frac{2}{3}ax^3 + C}{\frac{x^3}{3} + ax^2 + C}$$

$$= \frac{\frac{b^2}{4} + \frac{2}{3}ab}{\frac{b}{3} + a} = \frac{\frac{b^2}{12}\{3b + 8a\}}{\frac{1}{3}\{b + 3a\}} = \frac{b}{4}\left(\frac{3b+8a}{b+3a}\right)$$

for limits $x=0$ $x=c$ values

$$\bar{x} = \frac{c(8a+3c)}{4(3a+c)}$$

Let the parabola be $y^2 = 4mx$.



$\therefore b^2 = 4mx$

$\therefore AB = x = \frac{b^2}{4m}$.

Transfer the origin to B. and the eq^n becomes.

$$y^2 = 4m\left(x + \frac{b^2}{4m}\right) = 4mx + b^2.$$

$\therefore y^2 - b^2 = 4mx$.

Now let BC be the axis of x, and the eq^n becomes

$$x^2 - b^2 = 4my \text{ or } y = \frac{x^2 - b^2}{4m}.$$

$$\therefore \bar{x} = \frac{\int_a^b x(x^2-b^2)^2}{\int_a^b (x^2-b^2)^2} = \frac{\int_a^b x^5 - 2x^3b^2 + xb^4}{\int_a^b x^4 - 2x^2b^2 + b^4}$$

$$= \frac{\frac{1}{6}b^6 - \frac{1}{2}b^6 + \frac{1}{2}b^6}{\frac{1}{5}b^5 - \frac{2}{3}b^5 + b^5} = \frac{\frac{1}{6}b}{\frac{8}{15}} = \frac{5}{16}b$$

The curve is $y^2 = \frac{b^2}{a^2}(x^2 + a^2)$.

Make CB the axis of x and the tangent

$$x^2 = \frac{b^2}{a^2}(y^2 + a^2) \quad \text{or} \quad y^2 = \frac{a^2}{b^2}(x^2 + b^2).$$

$$\therefore \bar{x} = \frac{\int_0^b x(x^2 + b^2)}{\int_0^b (x^2 + b^2)} = \frac{\frac{b^4}{4} + \frac{b^4}{2}}{\frac{b^3}{3} + b^3}$$

$$= \frac{3}{4} \cdot b \times \frac{3}{4} = \frac{9}{16} b.$$

$$y = (a-x)\left(\frac{x}{a}\right)^{\frac{m}{n}}.$$

$$y^2 = (a-x)^2 \cdot \frac{x^{\frac{2m}{n}}}{a^{\frac{2m}{n}}}.$$

$$\therefore \bar{x} = \frac{\int x \left(a^2 - 2ax^2 + x^3\right) x^{\frac{2m}{n}}}{\int x \left(a^2 - 2ax + x^2\right) x^{\frac{2m}{n}}}$$

$$= \frac{\int x \; a^2 x^{\frac{2m+n}{n}} - 2a x^{\frac{2(m+n)}{n}} + x^{\frac{2m+3n}{n}}}{\int x \; a^2 x^{\frac{2m}{n}} - 2a x^{\frac{2m+n}{n}} + x^{\frac{2(m+n)}{n}}}$$

$$= \frac{\frac{1}{2(m+n)} a^2 x^{\frac{2(m+n)}{n}} - \frac{1}{2m+3n} \cdot 2a x^{\frac{2m+3n}{n}} + \frac{1}{2(m+2n)} x^{\frac{2(m+2n)}{n}}}{\frac{1}{2m+n} a^2 x^{\frac{2m+n}{n}} - \frac{1}{m+n} \cdot a x^{\frac{2(m+n)}{n}} + \frac{1}{2m+3n} \cdot x^{\frac{2m+3n}{n}}}$$

Take the integral from $x=0$ to $x=a$.

$$= a \left\{ \frac{\frac{1}{2m+2n} - \frac{2}{2m+3n} + \frac{1}{2m+4n}}{\frac{1}{2m+n} - \frac{1}{m+n} + \frac{1}{2m+3n}} \right\} = \frac{5a}{8} \quad \text{for } m = 2n = 0$$

Mean $\bar{x} = \dfrac{\int_0^{x'} lx^2 + \int_{x'}^{a} l'(ax - x^2)}{\int_0^{x'} lx + \int_{x'}^{a} l'(a-x)}$ where $x' = \dfrac{l'a}{l+l'}$

Now $\int_0^{x'} lx^2 = \frac{l}{3} \cdot x'^3 + C = \frac{l}{3} x'^3$

$\int_{x'}^{a} l'(ax - x^2) = -l'\left(a\frac{x^2}{2} - \frac{1}{3}x^3 + \frac{1}{6}a^3\right) = l'\left(\frac{1}{6}a^3 - \frac{1}{2}ax^2 + \frac{1}{3}x^3\right)$.

$\therefore$ numerator $= \frac{l'a^3}{6} - \frac{1}{2}l'a\left(\frac{l'a}{l+l'}\right)^2 + \frac{1}{3}(l+l')\left(\frac{l'a}{l+l'}\right)^3$

$= \frac{l'a^3}{6} - \frac{1}{6} \cdot l'a \cdot \left(\frac{l'a}{l+l'}\right)^2 = \frac{a^3 l'}{6}\left\{1 - \frac{l'^2}{(l+l')^2}\right\}$

$= \frac{a^3 l'}{6}\left\{\frac{l^2 + 2ll'}{(l+l')^2}\right\} = \frac{a^3 l' l}{6}\left\{\frac{l+2l'}{(l+l')^2}\right\}$.

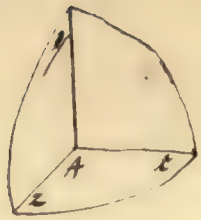
Similarly $\int_0^{x'} lx = \frac{1}{2}lx'^2$

$\int_{x'}^{a} l'(a-x) = l'\left(ax - \frac{x^2}{2}\right) + C = l'\left(\frac{a^2}{2} - a \cdot \frac{l'a}{l+l'} + \frac{(l'a)^2}{2(l+l')^2}\right)$

$\therefore$ denom. $= \frac{l'a^2}{2} - a \cdot \frac{l'a}{l+l'} + \frac{l'+l}{2} \cdot \frac{(l'a)^2}{(l+l')^2}$

$= \frac{a^2 l'}{2}\left\{1 - \frac{l'}{l+l'}\right\} = \frac{a^2 l' l}{2(l+l')}$

$\therefore \bar{x} = \frac{a}{3} \cdot \left(\frac{l+2l'}{l+l'}\right)$



For limits of x are $x=0$ and $x=a$.

For y are $y=0$ and $y=\sqrt{a^2-x^2-z^2}$ $\sqrt{a^2-x^2}$

For z are $z=0$ and $z=\sqrt{a^2-x^2-y^2}$.

$$\bar{x} = \frac{\int_0^a \int_0^{y'} \int_0^z x\,dx\,dy\,dz}{\int_0^a \int_0^{y'} \int_0^z 1\,dx\,dy\,dz} = \frac{\int_0^a \int_0^{y'} x\sqrt{a^2-x^2-y^2}\,dx\,dy}{\int_0^a \int_0^{y'} \sqrt{a^2-x^2-y^2}\,dx\,dy} = \frac{\int_0^a x\left(\frac{a^2-x^2}{2}\right)\frac{\pi}{4}\,dx}{\int_0^a \left(\frac{a^2-x^2}{2}\right)\frac{\pi}{4}\,dx}$$

$$= \frac{\frac{a^2x^2}{2} - \frac{x^4}{4} + C}{a^2x - \frac{2}{3}x^3 + C} = \frac{\frac{a^4}{4}}{\frac{2a^3}{3}} = \frac{3}{8}a.$$

Now $\int_0^z \int_0^{y'} z\,dz\,dy = \int_0^{y'} \frac{z^2}{2}\,dy = \int_0^{y'} \frac{y}{2}\sqrt{a^2-x^2-y^2}\,dy$

$$= \int_0^{y'} -\frac{1}{3}\left(a^2-x^2-y^2\right)^{\frac{3}{2}} + C = \int_0^{y'} -\frac{1}{3}\left(a^2-x^2\right)^{\frac{3}{2}}.$$

$$\int_0^a -\frac{1}{3}\left(a^2-x^2\right)\sqrt{a^2-x^2} = \frac{\pi}{16}a^4.$$

$$\int x^2\sqrt{a^2-x^2} = \frac{x^3}{3}\sqrt{a^2-x^2} + \frac{1}{3}\int \frac{-x^4}{\sqrt{a^2-x^2}} = \frac{x^3}{3}\sqrt{a^2-x^2} - \frac{1}{3}\int \frac{x^4(a^2-x^2)}{\sqrt{a^2-x^2}}$$

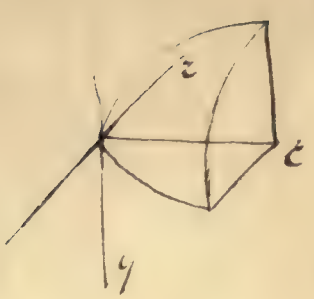
$$\int x^2\sqrt{a^2-x^2} = \frac{x^3}{4}\sqrt{a^2-x^2} - \frac{1}{4}\int \sqrt{a^2-x^2}\,dx + \frac{a^2}{8}\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$\int \sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}. \qquad \therefore \int_0^a -\frac{1}{3}\sqrt{a^2-x^2} = \frac{\pi a^4}{16}.$$

$$\therefore \bar{y} = \bar{z} = \bar{x} = \frac{3a}{8}.$$

$y^2 + z^2 = 4mx$.



die Umrisse sind $z = 0 \cdot z = \sqrt{4mx - y^2}$

$y = 0 \cdot y = 2\sqrt{mx}$.

$x = 0 \quad x = a$

$$\bar{x} = \frac{\int_0^a \int_y x}{\int_0^a \int_y 1} = \frac{\int_0^a \int_y x \sqrt{4mx - y^2}}{\int_0^a \int_y \sqrt{4mx - y^2}} = \frac{\int_0^a x \cdot 4mx^2 \frac{\pi}{4}}{\int_0^a 4mx \frac{\pi}{4}} = \frac{\frac{2}{3}a^3}{\frac{2}{3}a} = \frac{2}{3}a.$$

ähnlich $\bar{y} = \frac{\int_0^a \int_y y}{\frac{1}{2}ma^2} = \frac{2}{ma^2} \int_0^a \int_y y\sqrt{4mx - y^2} = \frac{2}{3ma^2} \int_0^a (4mx - y^2)^{\frac{3}{2}} + C$

$$= \frac{2}{3ma^2} \int_0^a 8 \cdot m^{\frac{3}{2}} \cdot x^{\frac{3}{2}} = \frac{2 \cdot 16 m^{\frac{1}{2}} a^{\frac{5}{2}}}{5\,3a^2} = \frac{32}{15} m^{\frac{1}{2}} a^{\frac{1}{2}} = \frac{16}{15} b.$$

$$\bar{z} = \frac{2}{ma^2} \int_0^a \int_y \int_z z = \frac{1}{ma^2} \int_0^a \int_y z^2 = \frac{1}{ma^2} \int_0^a \int_y 4mx - y^2$$

$$= \frac{1}{ma^2} \int_0^a \left(4mxy - \frac{1}{3}y^3\right) + C = 16m^2x^2 - \frac{64}{3} \cdot m^3x^3$$

$$= 4mx \cdot 2\sqrt{mx} - \frac{1}{3} \cdot 8 m^{\frac{3}{2}} x^{\frac{3}{2}} = \frac{16}{3} m^{\frac{3}{2}} x^{\frac{3}{2}}$$

$$= \frac{1}{ma^2} \left\{ \frac{16m^2a^3}{3} - \frac{16}{3} m^3a^4 \right) = \frac{16}{3} \cdot \{ma -$$

$$= \frac{1}{ma^2} \left\{ \frac{2}{3} 8 \cdot m^{\frac{3}{2}} x^{\frac{3}{2}} = \frac{16}{3} \right.$$

$$= \frac{1}{ma^2} \left\{ \frac{16}{3} \cdot \frac{2}{5} \cdot m^{\frac{3}{2}} a^{\frac{5}{2}} \right) = \frac{32}{15} m^{\frac{1}{2}} a^{\frac{1}{2}} = \frac{16}{5} b.$$

$$x^- = \frac{\int_0^a \int_0^b \int_0^{\sqrt{xy}} x \, dz}{\int_0^a \int_0^b \int_0^{\sqrt{xy}} dz} = \frac{\int_0^a \int_0^b x\sqrt{xy}}{\int_0^a \int_0^b \sqrt{xy}} = \frac{\frac{2}{3}\int_0^a x \cdot x^{\frac{1}{2}} b^{\frac{3}{2}}}{\frac{2}{3}\int_0^a x^{\frac{1}{2}} b^{\frac{3}{2}}} = \frac{\frac{2}{5}a^{\frac{5}{2}}}{\frac{2}{3}a^{\frac{3}{2}}} = \frac{3}{5}a$$

$$y^- = \frac{\int_0^a \int_0^b \int_0^{\sqrt{xy}} y}{\frac{4}{9}b^{\frac{3}{2}}a^{\frac{3}{2}}} = \frac{9}{4b^{\frac{3}{2}}a^{\frac{3}{2}}} \int_0^a \int_0^b y\sqrt{xy} = \frac{9}{2 \cdot 5} \int_0^a x^{\frac{1}{2}} \cdot b^{\frac{5}{2}} = \frac{9b^{\frac{5}{2}}}{2 \cdot 5 \cdot b^{\frac{3}{2}}a^{\frac{3}{2}}} \int_0^a x$$

$$= \frac{9}{2 \cdot 5} \cdot \frac{2}{3} \frac{b}{a^{\frac{3}{2}}} \cdot a^{\frac{3}{2}} = \frac{3}{5}b$$

$$z^- = \frac{9}{4b^{\frac{3}{2}}a^{\frac{3}{2}}} \int_0^a \int_0^b \int_0^{xy} z \cdot z = \frac{9}{8 \cdot b^{\frac{3}{2}}a^{\frac{3}{2}}} \int_0^a \int_0^b a^2 y$$

$$= \frac{9}{32 \cdot b^{\frac{3}{2}}a^{\frac{3}{2}}} \cdot a^2 b^2 = \frac{9}{32} a^{\frac{1}{2}} b^{\frac{1}{2}}$$

All the C.G. of gravs. will evidently be in the axis of x.

$$\therefore \bar{y} = 0.$$

$$\bar{x} = \frac{\int_0^{2a}\int_0^{y}\int_{\frac{x^2}{4a}}^{\beta x}\, x\, dx\, dy\, d\beta}{\int_0^{2a}\int_0^{y}\int_{\frac{x^2}{4a}}^{\beta x}\, dx\, dy\, d\beta} = \frac{\int_0^{2a}\int_0^{y}\left(\beta+\beta'\right)x^2\, dy\, dx}{\int_0^{2a}\int_0^{y}\left(\beta+\beta'\right)x\, dy\, dx} = \frac{\int_0^{2a} x^2\sqrt{2ax-x^2}\, dx}{\int_0^{2a} x\sqrt{2ax-x^2}\, dx}$$

$$\int x^2\sqrt{2ax-x^2}\, dx = \int x\cdot\frac{2ax^2-x^3}{\sqrt{2ax-x^2}}$$

$$\int \frac{x^4}{\sqrt{2ax-x^2}} = x^3\frac{\sqrt{2ax-x^2}}{4} - \frac{7}{4}\int \frac{x^3}{\sqrt{2ax-x^2}}$$

$$\int \frac{x^3}{\sqrt{2ax-x^2}} = -\frac{1}{4}\left\{\frac{x^2}{3}+\frac{5ax}{3.2}+\frac{5.3a^2}{3.2.1}\right\}\sqrt{2ax-x^2} + \frac{5}{8}a^3\cos^{-1}\frac{a-x}{a}$$

$$\therefore \int_0^{2a} x^2\sqrt{2ax-x^2} = \frac{5}{8}a^4\pi.$$

$$\text{and} \int_0^{2a} x\sqrt{2ax-x^2} = -\frac{1}{3}\left\{\frac{x}{2}+\frac{3a}{2.1}\right\}\sqrt{2ax-x^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-x}{a}\right)+C$$

$$= \frac{1}{2}a^2\pi$$

$$\therefore \bar{x} = \frac{\frac{5}{8}a^4\pi}{\frac{1}{2}a^2\pi} = \frac{5}{4}a.$$

$$\bar{z} = \frac{2}{(\beta+\beta')a^3\pi}\int_0^{2a}\int_0^{y}\int_{-z}^{z} z\, dz\, dy\, dx = \frac{1}{(\beta+\beta')a^3\pi}\int_0^{2a}\int_0^{y}(\beta+\beta)z^2\, dy\, dx$$

$$= \frac{(\beta+\beta')}{a^3\pi}\int_0^{2a} x^2\sqrt{2ax-x^2}\, dx = \frac{\beta+\beta'}{a^3\pi}\cdot\frac{5}{8}a^4\pi = \frac{5a}{8}(\beta+\beta')$$

Into ⊙ with the axis for ch. $y = \sqrt{a^2 - x^2}$.

$$\therefore ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{a}{\sqrt{a^2 - x^2}}$$

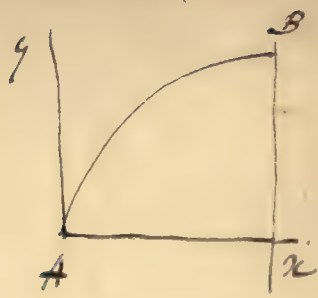
$$\therefore \bar{x} = \frac{\int x\, ds}{\int ds} = \frac{-a\sqrt{a^2 - x^2} + c}{a \sin^{-1}\frac{x}{a} + c} = \frac{-ay + c}{a \sin^{-1}\frac{x}{a} + c}$$

If the axis of x bisects the arc, and c be the chord of the arc we have

$$x = \frac{-ac}{-a\left(\frac{\pi}{2} + \sin^{-1}\sqrt{\frac{b^2 - a^2}{a}}\right)}$$

$$= \frac{ac}{r}$$

$$y = 0.$$



& the origin & y. axes of x. y.

$$\bar{x} = \frac{\int_0^{2a} x \cdot x \, dy}{\int_0^{2a} x \, dy} = \frac{\int_0^{2a} x \cdot \sqrt{2ax}}{\int_0^{2a} x \cdot \frac{\sqrt{2a}}{\sqrt{x}}} = \frac{\frac{2a}{5}(2a)^{\frac{5}{2}}}{2\sqrt{2a}(2a)^{\frac{1}{2}}} = \frac{\frac{2}{3}(2a)^{\frac{1}{2}}\sqrt{2a}}{\sqrt{2a}\cdot 2\sqrt{2a}}$$

$$= \frac{2a}{3}.$$

$$\bar{y} = \frac{\int_0^{2a} y \cdot dx}{\int_0^{2a} y \, dx} = \frac{\int_0^{2a} \left(a \sin^{-1}\frac{x}{a} + \sqrt{2ax - x^2}\right)\sqrt{\frac{2a}{x}}}{2\sqrt{2a}(2a)^{\frac{1}{2}}} = \frac{\int_0^{2a} x \cdot \left(\frac{a}{\sqrt{x}}\sin^{-1}\frac{x}{a} + \sqrt{2a-x}\right)}{2\sqrt{2a}}$$

$$\int \frac{a}{\sqrt{x}} \sin^{-1}\frac{x}{a} = 2a\sqrt{x}\sin^{-1}\frac{x}{a} - 2a\int_x \frac{\sqrt{x}}{\sqrt{2ax - x^2}}$$

$$= 2a\sqrt{x}\sin^{-1}\frac{x}{a} - 2a\cdot\int_x \frac{1}{\sqrt{2a-x}}$$

$$= 2a\sqrt{x}\sin^{-1}\frac{x}{a} + 4a\sqrt{2a-x} + C.$$

$$\int \sqrt{2a-x} = -\frac{2}{3}(2a-x)^{\frac{3}{2}} + C$$

$$\therefore \int_x \left(\frac{a}{\sqrt{x}}\sin^{-1}\frac{x}{a} + \sqrt{2a-x}\right) = 2a\sqrt{x}\cdot\sin^{-1}\frac{x}{a} + 4a\sqrt{2a-x} - \frac{2}{3}(2a-x)^{\frac{3}{2}} + C.$$

$$\therefore \int_0^{2a} \left(\frac{a}{\sqrt{x}}\sin^{-1}\frac{x}{a} + \sqrt{2a-x}\right) = \left(2a\right)^{\frac{3}{2}}\pi - 4a\sqrt{2a} + \frac{2}{3}(2a)^{\frac{3}{2}}$$

$$= \left(2a^{\frac{3}{2}}\right)\left(\pi - \frac{4}{3}\right) =$$

$$\therefore \bar{y} = \frac{(2a)^{\frac{3}{2}}\left(\pi - \frac{4}{3}\right)}{2a\sqrt{2a}} = a\left(\pi - \frac{4}{3}\right)$$

$$y = \frac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}$$

$$\frac{ds}{dx} = \sqrt{1 + \frac{1}{4}\left(e^{\frac{2x}{a}} - 2 + e^{-\frac{2x}{a}}\right)} = \frac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

$$y\frac{ds}{dx} = \frac{1}{4}a\cdot\left\{e^{\frac{2x}{a}} + 2 + e^{-\frac{2x}{a}}\right\}$$

$$\bar{x} = \int x\, y\, \frac{ds}{dx}\, dx \Big/ \int y\, \frac{ds}{dx}\, dx$$

$$\int_0^c y\frac{ds}{dx}\,dx = \frac{1}{2}a\left\{\quad\right\}$$

$$\int y = xy - \int x\, dy$$

$$= xy - \int x$$

$$= xy - \int_0^c y\, dx$$

In this example the limits of x are $x=0$ $x=a$.

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{m^2}{x}} = \frac{\sqrt{x+m}}{\sqrt{x}}$$

$$\therefore \bar{x} = \frac{\int_0^a x\,ds}{\int_0^a ds} = \frac{\int_0^a \sqrt{x^2+mx}}{\int_0^a \frac{x+\frac{m}{2}}{\sqrt{x^2+mx}} + \frac{m}{2}\int_0^a \frac{1}{\sqrt{x^2+mx}}}$$

$$\int x\sqrt{x^2+mx} = \int x\sqrt{\left(x+\tfrac{m}{2}\right)^2 - \frac{m^2}{4}} = \frac{x+\frac{m}{2}}{2}\sqrt{x^2+mx} - \frac{m^2}{8}\log_e\left\{x+\frac{m}{2}+\sqrt{x^2+mx}\right\}+C.$$

$$\int_0^{2m} x\sqrt{x^2+mx} = \frac{3m^2}{4}\sqrt{2} - \frac{m^2}{8}\log_e\left\{3+\frac{2m}{m}\sqrt{2}\right\}.$$

$$= \frac{m^2}{8}\left\{6\sqrt{2} - \log_e\left(3+2\sqrt{2}\right)\right\}.$$

$$\int \frac{x+\frac{m}{2}}{\sqrt{x^2+mx}} + \frac{m}{2}\int \frac{1}{\sqrt{x^2+mx}} = \sqrt{x^2+mx} + \frac{m}{2}\log_e\left\{x+\frac{m}{2}+\sqrt{x^2+mx}\right\}.$$

$$\therefore \int_0^{2m} \frac{x+\frac{m}{2}}{\sqrt{x^2+mx}} + \frac{m}{2}\int_0^{2m} \frac{1}{\sqrt{x^2+mx}} = m\sqrt{2} + \frac{m}{2}\log_e\left\{3+\sqrt{2}\right\}.$$

$$\therefore \bar{x} = \frac{m}{4}\left\{\frac{6\sqrt{2} - \log_e\left(3+2\sqrt{2}\right)}{2\sqrt{2}+\log_e\left(3+2\sqrt{2}\right)}\right.$$

$$= \frac{m}{4}\left\{\frac{6\sqrt{2} - \log_e\left(1+\sqrt{2}\right)^2}{2\sqrt{2}+\log_e\left(1+\sqrt{2}\right)^2}\right\} = \frac{m}{4}\cdot\frac{3\sqrt{2}-\log_e\left(1+\sqrt{2}\right)}{\sqrt{2}+\log_e\left(1+\sqrt{2}\right)}$$

$$\bar{y} = 0.$$

? on a curve of double curvature we have.

$$ds = \sqrt{1+(dy)^2+(dz)^2}.$$

Now the equations to the helix are $y^2+x^2 = a^2$

$$z = b \cos^{-1}\frac{x}{a}.$$

$$\therefore (dy)^2 = \frac{x}{a^2-x^2}. \quad (dz)^2 = \frac{b^2}{a^2-x^2}$$

$$\therefore ds = \sqrt{\frac{a^2+b^2}{\sqrt{a^2-x^2}}}$$

$$\therefore \bar{x} = \frac{\int_x^{x^2} x\, ds}{\int_x ds} = \frac{\int x \frac{-x}{\sqrt{a^2-x^2}}}{\int_x \frac{-1}{\sqrt{a^2-x^2}}} = \frac{\sqrt{a^2-x^2}}{\cos^{-1}\frac{x}{a}} = \frac{yb}{z}.$$

$$\bar{y} = \frac{\int_x^{x^2} y \cdot ds}{\int_x ds} = \frac{\int_x^{x^2} -1}{\int_x^{x^2} \frac{-1}{\sqrt{a^2-x^2}}} = \frac{(a-x)}{\cos^{-1}\frac{x}{a}} = \frac{b(a-x)}{z}.$$

$$\bar{z} = \frac{\int_x z \cdot ds}{\int_x ds} = \frac{\int_x \frac{b \cos^{-1}\frac{x}{a}}{\sqrt{a^2-x^2}}}{\int_x ds} = \frac{\frac{b}{2}\cdot\left(\cos^{-1}\frac{x}{a}\right)^2}{\cos^{-1}\frac{x}{a}} = \frac{b}{2}\cos^{-1}\frac{x}{a}$$

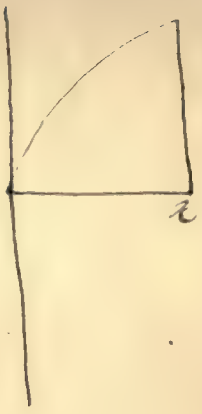
$$= \frac{z}{2}.$$

$$\therefore \bar{x} = \frac{by}{z}, \quad \bar{y} = \frac{b(a-x)}{z}, \quad \bar{z} = \frac{z}{2}.$$

$$y = ax.$$

$$\frac{dy}{dx} = a \quad \therefore \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+a^2}$$

$$\therefore \bar{x} = \frac{\int_0^b xy \cdot ds}{\int_0^b y\, ds} = \frac{\int_0^b ax^2}{\int_0^b ax} = \frac{\frac{2}{3}\frac{ax^3}{a}}{x^2} = \frac{2}{3}b.$$



$2a$ = corde cissoïde $y = a \operatorname{vers}^{-1}\frac{x}{a} + \sqrt{2ax - x^2}$

tangente $\frac{dy}{dx} = \sqrt{\frac{2a-x}{x}}$. $\frac{dx}{x} = \sqrt{\frac{2a}{x}}$

$k' = \frac{\int_0^{2a} \sqrt{\frac{2a}{x}} \cdot \sqrt{x}\left(a \operatorname{vers}^{-1}\frac{x}{a} + \sqrt{2ax-x^2}\right)}{\int_0^{2a} \sqrt{\frac{2a}{x}} \cdot \sqrt{x}\left(a \operatorname{vers}^{-1}\frac{x}{a} + \sqrt{2ax-x^2}\right)}$

$\int a\sqrt{x} \operatorname{vers}^{-1}\frac{x}{a} = \frac{2a}{3} x^{\frac{3}{2}} \operatorname{vers}^{-1}\frac{x}{a} - \frac{2a}{3}\int \frac{x^{\frac{3}{2}}}{\sqrt{2ax-x^2}}$

$= \frac{2a}{3} x^{\frac{3}{2}} \operatorname{vers}^{-1}\frac{x}{a} + \frac{2a}{3}\int \sqrt{2a-x} - \frac{4a^2}{3}\int \frac{1}{\sqrt{2a-x}}$

$\therefore \int_0^{2a} a\sqrt{x} \operatorname{vers}^{-1}\frac{x}{a} = \frac{(2a)^{\frac{5}{2}}\pi}{3} + \frac{2}{9}(2a)^{\frac{5}{2}} - \frac{2}{3}(2a)^{\frac{5}{2}}$

$= 2a^{\frac{5}{2}}\left\{\frac{\pi}{3} - \frac{4}{9}\right\}$

$\int \sqrt{x}\sqrt{2ax-x^2} = \int x\sqrt{2a-x} = -\int (2a-x)^{\frac{3}{2}} + 2a\int (2a-x)^{\frac{1}{2}}$

$\therefore \int_0^{2a} \sqrt{x}\sqrt{2ax-x^2} = -\frac{2}{5}(2a)^{\frac{5}{2}} + \frac{2}{3}(2a)^{\frac{5}{2}} = \frac{4}{15}(2a)^{\frac{5}{2}}$

$\therefore$ Numérateur $= (2a)^{\frac{5}{2}}\left\{\frac{\pi}{3} - \frac{4}{9} + \frac{4}{15}\right\} \times \sqrt{2a}$.

$\int \frac{a}{\sqrt{x}} \operatorname{vers}^{-1}\frac{x}{a} = 2a\sqrt{x} \operatorname{vers}^{-1}\frac{x}{a} - 2a\int \frac{1}{\sqrt{2a-x}}$

$\int_0^{2a} \frac{a}{\sqrt{x}} \operatorname{vers}^{-1}\frac{x}{a} = (2a)^{\frac{3}{2}}\pi - \frac{8}{3}(2a)^{\frac{3}{2}}$

$\int_0^{2a} \sqrt{2a-x} = \frac{2}{3}(2a)^{\frac{3}{2}}$

$\therefore$ dénom. $= (2a)^{\frac{3}{2}}\left\{\pi - \frac{4}{3}\right\}\sqrt{2a}$.

$\therefore k' = \frac{2a \cdot \left(\frac{\pi}{3} - \frac{4}{9} + \frac{4}{15}\right)}{\pi - \frac{4}{3}} = 2a\left\{\frac{45\pi - 24}{45(3\pi - 4)}\right\} = \frac{2a}{15}\left\{\frac{15\pi - 8}{3\pi - 4}\right\}$

Kedua-dua kaitan parabola.

$y = 2\sqrt{mx}$. $\frac{dy}{dx} = \sqrt{\frac{m}{x}}$. $\frac{ds}{dx} = \sqrt{\frac{x+m}{x}}$.

$$\therefore \bar{x} = \frac{\int_a^b y \cdot \frac{ds}{dx}}{\int_a^b y \cdot \frac{ds}{dx}} = \frac{\int_a^b x\sqrt{x+m}}{\int_a^b \sqrt{x+m}}$$

$$\int_a^b x\sqrt{x+m} = \int_a^b (x+m)^{\frac{3}{2}} - \int_a^b m(x+m)^{\frac{1}{2}}.$$

$$= \frac{2}{5}(x+m)^{\frac{5}{2}} - \frac{2m}{3}\cdot(x+m)^{\frac{3}{2}} + C.$$

$$\therefore \int_a^b x\sqrt{x+m} = \left(\frac{2}{5}(b+m) - \frac{2m}{3}\right)(b+m)^{\frac{3}{2}} - \frac{2}{5}m^{\frac{5}{2}} + \frac{2}{3}m^{\frac{5}{2}}.$$

$$= \left(\frac{2}{5}b - \frac{4m}{15}\right)(b+m)^{\frac{3}{2}} + \frac{4}{15}m^{\frac{5}{2}}.$$

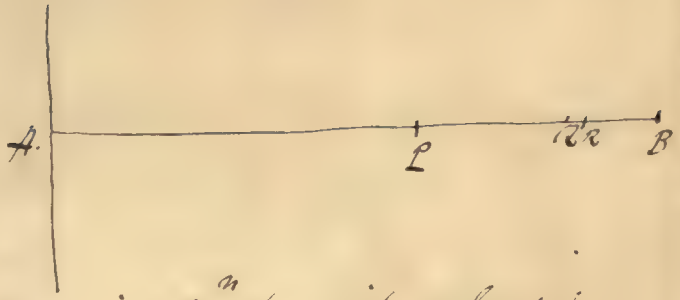
$$\text{Also } \int_a^b \sqrt{x+m} = \frac{2}{3}\left\{(b+m)^{\frac{3}{2}} - m^{\frac{3}{2}}\right\}.$$

$$\therefore \bar{x} = \frac{\frac{2}{15}\cdot\left\{3b - 2m\left\{(b+m)^{\frac{3}{2}}\right\} + 2m^{\frac{5}{2}}\right\}}{\frac{2}{3}\left\{(b+m)^{\frac{3}{2}} - m^{\frac{3}{2}}\right\}}$$

$$= \frac{1}{5}\cdot\left(3b - 2m\left\{\frac{(b+m)^{\frac{3}{2}} + 2m^{\frac{5}{2}}}{(b+m)^{\frac{3}{2}} - m^{\frac{3}{2}}}\right\}\right) \times$$

$$= \frac{1}{5}\,\frac{(3b-2m)(b+m)^{\frac{3}{2}} + 2m^{\frac{5}{2}}}{(b+m)^{\frac{3}{2}} - m^{\frac{3}{2}}}.$$

A is point from wh. the density is computed. AB the given line $AL = a$. $AB = b$. $AK = x$.

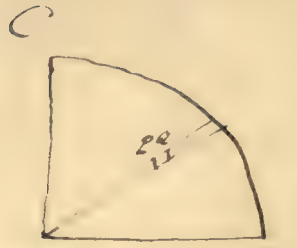


$\therefore$ an element of the line length dx. $\therefore x^n \cdot dx$ = its volume

$\therefore$ its moment $x^{n+1} dx$ about A.

$$\therefore \bar{x} = \frac{\int_a^b x^{n+1}}{\int_a^b x^n} = \frac{n+1}{n+2} \cdot \frac{b^{n+2} - a^{n+2}}{b^{n+1} - a^{n+1}}.$$

Take origin at C.



For an elementary particle = $dr.d\theta$

$\therefore \quad r^{n+2} dr\, d\theta \cos\theta$ = moment of particle about C.

$$\therefore \quad \bar{x} = \frac{\int_0^a \int_0^{\frac{\pi}{2}} r^{n+2} \cos\theta}{\int_0^a \int_0^{\frac{\pi}{2}} r^{n+1}\, r\, d\theta} = \frac{(n+2)a}{n+3} \cdot \frac{\int_0^{\frac{\pi}{2}} \cos\theta}{\int_0^{\frac{\pi}{2}} 1} = \frac{(n+2)a}{n+3} \cdot \frac{\sin\theta + C}{\theta + C}$$

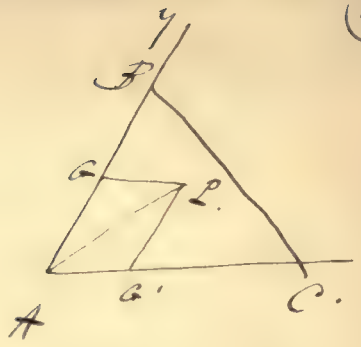
$$= \frac{2a}{\pi} \cdot \left(\frac{n+2}{n+3} \right)$$

Also we have

$$\bar{y} = \frac{\int_0^a \int_0^{\frac{\pi}{2}} r^{n+2} \sin\theta\, d\theta}{2(n+2) \cdot \frac{r^{n+2}}{n+2}\, a^{n+2}\, \pi} = \frac{2(n+2)}{\pi a^{n+2}} \frac{a^{n+3}}{(n+3)} \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta$$

$$= \frac{2(n+2)}{n+3} \cdot \frac{a}{\pi} \cdot -\cos\theta + C.$$

$$= \frac{2(n+2)}{n+3} \cdot \frac{a}{\pi} \quad \text{as for } \bar{x}$$



Take L as origin & LA, LB as co-ord axes.

$$x = \frac{\frac{a}{l}u}{\frac{u}{a}+\frac{u}{b}+\frac{u}{c}} = \frac{a}{b}\left(\frac{abc}{ab+bc+ac}\right) = \text{const} = LG$$

$$y = \frac{\frac{b}{c}u}{\frac{u}{a}+\frac{u}{b}+\frac{u}{c}} = \frac{b}{c}\cdot\frac{abc}{ab+bc+ac} = \text{const} = LG'$$

$\therefore$ L is the centroid of $\triangle ABC$.

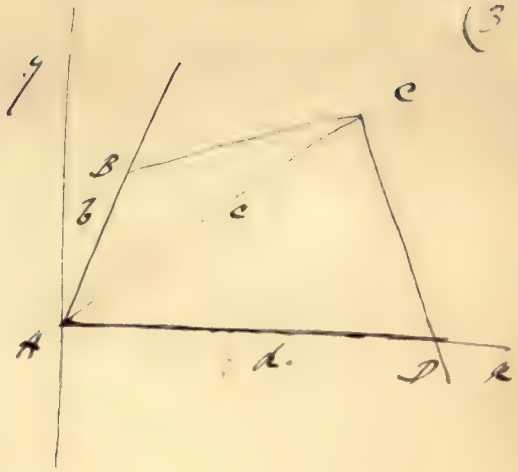
Also $AL = f$.

$$\therefore f^2 = x^2 + y^2 + 2xy\cos A.$$

$$= \left(\frac{c^2}{b^2}+\frac{b^2}{c^2}+2\cos A\right)\left(\frac{abc}{ab+bc+ac}\right)^2.$$

$$= \left(\frac{c^4+b^4+2c^2b^2\cos A}{b^2c^2}\right)\cdot\left(\frac{abc}{ab+bc+ac}\right)^2$$

$$= a\cdot\left(\frac{c^4+b^4+2c^2b^2\cos A}{ab+bc+ac}\right)^{\frac{1}{2}}$$



Take y & x as before cont^{d}.

$$\bar{x} = \frac{\Sigma(Px)}{\Sigma P}$$

$$= \frac{Pb\cos(b,x) + Cc\cos(c,x)R + dR}{P+R+R.}$$

$$\bar{y} = \frac{Pb\sin(b,x) + Cc\sin(c,x)R}{P+R+R.}$$

Let ρ = dist of the cen. of fcs fm A.

$$\therefore \rho^2 = \bar{x}^2 + \bar{y}^2$$

$$(P+R+R)^2 = P^2b^2\cos^2(b,x) + 2Pb\cos(b,x)\{RC\cos(c,x) + dR\}$$
$$+ R^2c^2\cos^2(c,x) + 2cx(2R\cos(c,x)$$
$$+ R^2x^2.$$

$$+ P^2b^2\sin^2(b,x) + 2P.Rcb.\sin(b,x)\sin(c,x) + R^2c^2\sin^2(c,x).$$

$$= P^2b^2 + R^2c^2 + R^2d^2 + 2PR^2cd.\cos(b,c) +$$

$$+ 2PR.bd.\cos(b,x) + 2R^2R.cx\cos(c,x).$$

Let f, f' be two magnitudes of p''

Then $\bar{x} = \dfrac{ap + a'p' + a''f(p'')}{p + p' + f(p'')}$ (1)

$$\bar{y} = \frac{bp + b'p' + b''f(p'')}{p + p' + f(p'')} \quad (2)$$

From (2) $\bar{y}\{(p + p' + f(p'')\} = bp + b'p' + b''f(p'')$

$$\therefore (\bar{y} - b'')f(p'') = bp + b'p' - \bar{y}(p + p')$$

$$\therefore f(p'') = \frac{bp + b'p' - \bar{y}(p + p')}{\bar{y} - b''}$$

Substituting in (1)

$$\bar{x} = \frac{ap + a'p' + a''\left\{\dfrac{bp + b'p' - \bar{y}(p + p')}{\bar{y} - b''}\right\}}{p + p' + \dfrac{bp + b'p' - \bar{y}(p + p')}{\bar{y} - b''}}$$

$$= \frac{(ap + a'p' - a''p - a''p')\bar{y} - b''(ap + a'p') + a''(bp - bp')}{-(p + p')b'' + bp + b'p'}$$

$$\therefore \bar{x}\{(b - b'')p + (b' - b'')p'\} + b''(ap + a'p') = \bar{y}\{(a - a'')p + (a' - a'')p'\} + a''(bp - bp')$$

$$\left.\begin{array}{l}\text{the eq}^{\text{n}}\text{ of}\\ \text{the req}^{\text{d}}\end{array}\right\} =$$

3.)

length of the path described by c of $\mathcal{J} = 2\pi b$.

area of the $\mathcal{J} = a^2\pi$.

$\therefore V = 2\pi b \times a^2\pi = 2\pi^2 a^2 b$.

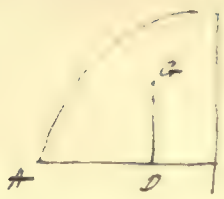
also circumf. of $\mathcal{J}$ whose rad = a is $2a\pi$

$\therefore S = 2\pi b \times 2a\pi = 4\pi^2 ab$.

4.) area of the $\mathcal{J} = \pi ab$.

length of path described by c of $\mathcal{J} = 2c\pi$

$\therefore V = \pi^2 abc$.



$\bar{y}$ of $\frac{1}{4}$ circle $\bar{y} = \frac{\int \frac{1}{2} y \, dy}{\int dy} = \frac{\frac{1}{2}\int_0^a y^2}{\int_0^a y}$

$a^2 \int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta = a^2 \int_0^{\frac{\pi}{2}} (1-\cos^2\theta)^2 \, d\theta = a^2 \int_0^{\frac{\pi}{2}} (1 - 2\cos^2\theta + \cos^4\theta) \, d\theta$

$\int \theta^2 \cos\theta = -\theta^2 \sin\theta + 2\int \theta \sin\theta$ ——— $\int \theta \sin\theta = \theta(-\cos\theta) - \int -\cos\theta = \theta(-\cos\theta) + \sin\theta$

$= -\theta^2 \sin\theta + 2\theta\cos\theta + 2\sin\theta.$

$\int \cos 2\theta \, d\theta = \int \frac{1}{2}(1+\cos 2\theta) = \frac{\theta}{2} + \frac{1}{2}\cdot\frac{\sin 2\theta}{2} = \frac{\theta}{2} + \frac{\sin 2\theta}{4}$

$\int \cos^2\theta = \int (1-\sin^2\theta)\cos\theta \, d\theta = \sin\theta - \frac{1}{3}\sin^3\theta.$

$\therefore \int (\theta^2 \cos^2\theta + 2\theta\cos^2\theta + \cos^4\theta) = -\theta^2\sin\theta + 2\theta\cos\theta + 2\sin\theta + \frac{\theta^2}{2} - \frac{\theta\sin 2\theta}{2} - \frac{\cos 2\theta}{4} - \sin\theta + \frac{1}{3}\sin^3\theta + C$

$= 2 - \frac{1}{4} - 1 + \frac{1}{3} = \frac{13}{12} + C \quad$ when $\theta = 0 \quad \therefore C = -\frac{13}{12}$.

$= \left(\pi^2 - 2 + \frac{\pi^2}{2} - \frac{1}{4} + 1 - \frac{1}{3} - \frac{13}{12}\right) = \frac{3\pi^2}{2} - \frac{8}{3}$

$\therefore$ the numerator of the first part $= \frac{a^3}{2}\left(\frac{3\pi^2}{2} - \frac{8}{3}\right)$. $\therefore$ denom $= \frac{3}{2}\frac{a^2\pi}{4}$

$\therefore \bar{y} = a\left(\frac{\pi}{2} - \frac{8}{9\pi}\right) = \bar{x}$.

from the axis the semi cylinder $= \frac{3}{2}\pi a^2$.

also length of path described by c. of gr. $= 2a\pi\left(\frac{\pi}{2} - \frac{8}{9\pi}\right)$

$\therefore V = \frac{3}{2}\pi a^2\left(\pi^2 a - \frac{16a}{9}\right) = \frac{a^3\pi}{2}\left(3\pi^2 - \frac{16}{3}\right)$.

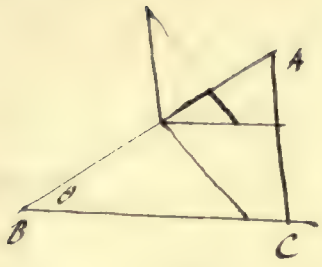
$= \pi a^3\left(\frac{3}{2}\pi^2 - \frac{8}{3}\right)$

Ans for the surface semi cylinder $y = a\left(\pi - \frac{4}{3}\right)$.

$\therefore$ path described by c. of gr. $= 2a\pi\left(\pi - \frac{4}{3}\right)$

Area surface semi cylinder $= 4a$

$\therefore S = 4a \times 2a\pi\left(\pi - \frac{4}{3}\right) = 8a^2\pi\left(\pi - \frac{4}{3}\right)$.



Let θ = req'd angle of incl.

Resolve forces vertically

$\frac{W}{3} +$

Resolve forces // to AB.

$$\therefore \frac{W}{3}\sin\theta - \frac{W}{2}\sin\theta + \frac{W}{3} + \frac{W}{3}\cos\theta = 0.$$

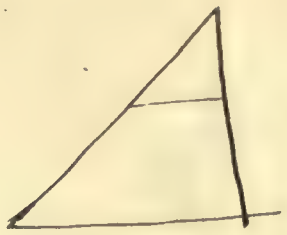
$$+\frac{2}{3}\sin\theta = \frac{1}{3}\{1+\cos\theta\}.$$

$$\frac{4}{3}\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2}{3}\{2\cos^2\frac{\theta}{2}$$

$$2\tan\frac{\theta}{2} = \cos\frac{\theta}{2}.$$

$$\tan\frac{\theta}{2} = \frac{1}{2}$$

$$\text{or } \theta = 2\tan^{-1}\frac{1}{2}.$$



Let this problem of the Ladder be illustrated by P.

Then $\frac{P}{W} = \tan\theta$

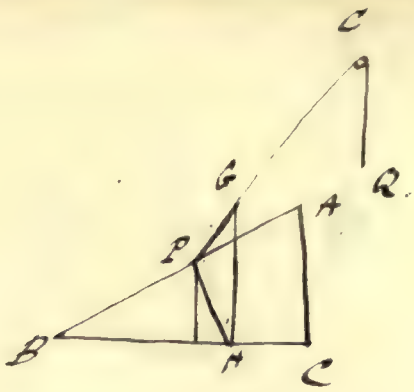
Next let W be illustrated by R.

Then $\frac{R}{W} = \sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{P}{\sqrt{W^2+P^2}}$

$\therefore R^2(W^2+P^2) = P^2W^2$

$W^2(R^2-P^2) = -R^2P^2$

$\therefore W = \frac{RP}{\sqrt{P^2-Q^2}}$



The tension of the string must be the same throughout since the pulley is smooth and since R is the tension of the portion of the string CR, it must also be the tension of CP.

Therefore if $\angle CPH = \beta$, and $\angle HBC = \theta$

we have (Trouble's Rule)

$$\frac{P}{R} = \frac{\cos\beta}{\sin\theta}$$

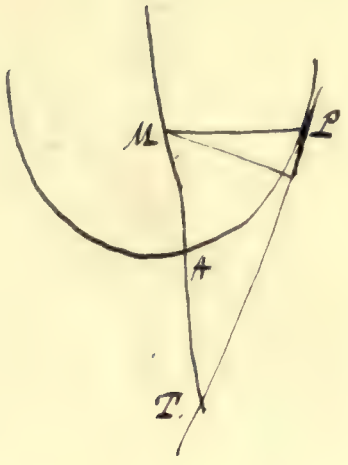
Let R be the pressure on the inclined plane.

$$\therefore \frac{R}{R} = \frac{PH}{GH} = \frac{\sin PGH}{\cos\beta} = \frac{\sin\left(\frac{\pi}{2} - \beta - \theta\right)}{\cos\beta}$$

$$= \frac{\cos(\theta+\beta)}{\cos\beta}$$

A the origin $AM = x$. $ML = y$

Coord of L.



Then $\tan PTM = \frac{dy}{dx} = \sqrt{\frac{a}{x}}$.

Resolve forces ⊥ tangent.

$\therefore \mu y \sin PTM = W \cos PTM$.

or $\tan PTM = \frac{W}{\mu y} = \sqrt{\frac{a}{x}}$.

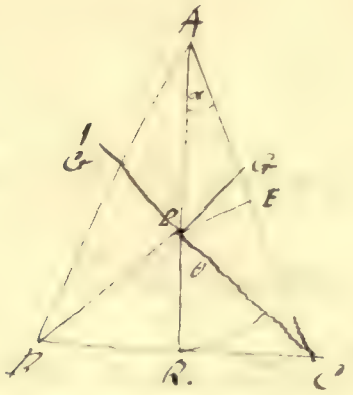
$\therefore W^2 x = \mu^2 y^2 a$.

$= 4\mu^2 a^2 x$

$\therefore W = 2\mu a$.

or $\frac{W}{\mu} = 2a$. $= \frac{1}{2}$ Latus rectum

of this property holds the particle will rest in any position, if not it will not rest in any position



The attraction of BC upon P =

$$\int_0^\theta \frac{2\mu b}{b^2}\cos\theta = \frac{2\mu}{b}\sin\theta \text{ in } PR$$

$$\int_0^\theta \frac{2\mu b}{b}\,d\theta = \frac{2\mu}{b}\sin\theta \text{ in } RC$$

$$\text{or } \frac{\mu}{b}\sin\theta \text{ in } RB.$$

The former alone is wanted

Join EF which bisects the $\angle APC$ $\therefore \angle FPC = \frac{1}{2}(\pi-\theta)$

Also if PE be $\perp$ to AC. $PE = a\sin\alpha$

$\therefore$ attract. of AC in direction PC $= \dfrac{\mu\sin\frac{1}{2}(\pi-\theta)}{a\sin\alpha}$

= att. of AB in direct. PG

$\therefore$ Whole part of the attraction of AB. and AC in direct. AP

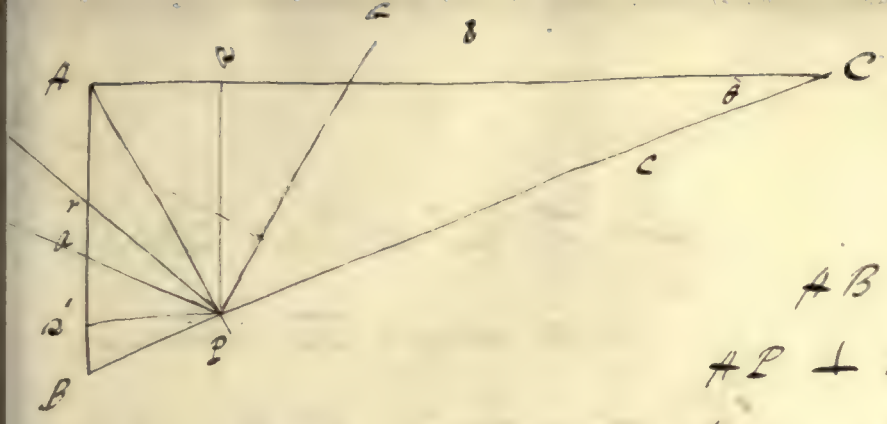
$$= \frac{2\mu\sin\frac{1}{2}(\pi-\theta)\cos\frac{1}{2}(\pi-\theta)}{a\sin\alpha}$$

$$= \frac{2\mu\sin\theta}{a\sin\alpha}$$

= attract. of BC in direct. PR.

$$= \frac{2\mu\sin\theta}{b}$$

$\therefore \sin\alpha = \dfrac{b}{a}$ or vertical angle $= 2\sin^{-1}\left(\dfrac{b}{a}\right)$



$AB = a$. $AC = b$ the sides of the Δ

$AP \perp$ to BC $\therefore$ P the position of the particle Pc, Pc' bisecting the angles APC, APB. $\therefore \angle cPC = \angle c'PB = \frac{\pi}{4}$

Then the whole of AC's attract. in direction $Pc = \frac{\mu \Delta \frac{\pi}{4}}{Pc}$

$Pc = AP \cos\theta = AB \cos^2\theta = \frac{ab^2}{c^2}$ $= \frac{\mu}{\sqrt{2}} \cdot \frac{c^2}{ab^2}$.

also whole of AB's att. in direct. $Pc' = \frac{\mu \Delta \frac{\pi}{4}}{Pc'}$

$= \frac{\mu}{\sqrt{2}} \cdot \frac{c^2}{a^2 b}$.

Let PP' be the direction & mag. the resultant of these attractions acting $\angle$ $P'Pc' = \phi$.

Then $\tan\phi = \frac{\text{att}^{\text{n}} \text{ acting in direction } Pc}{\text{att}^{\text{n}} \text{ acting in direction } Pc'} = \frac{\frac{1}{ab^2}}{\frac{1}{a^2 b}} = \frac{a}{b}$

$= \tan\theta$.

Also $\angle\ Pcx' = \frac{\pi}{2} - 45 + \theta = 45 + \theta$.

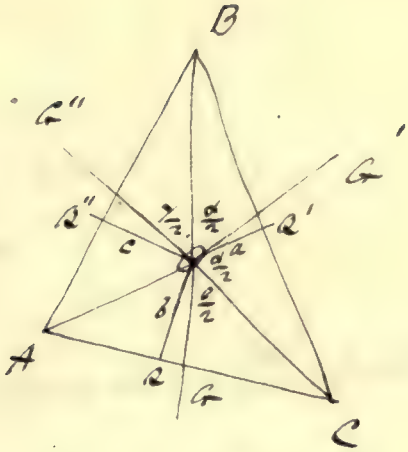
Also $\angle Pcx = \angle Pxa + \phi = \angle Pxa + \theta$

$\therefore \angle Pxa = 45^\circ$.

Therefore the resultant attn. acts $\perp$ to AB at an angle of 45°.

Also $r^2 = \frac{\mu^2}{2}\left\{\frac{c^4}{a^4b^2} + \frac{c^4}{a^2b^4}\right\} = \frac{\mu^2}{2}\left\{c^4\left\{\frac{a^2+b^2}{b^4a^4}\right\}\right\} = \frac{\mu^2}{2} \cdot \frac{c^6}{a^4b^4}$

$\therefore r = \frac{\mu}{\sqrt{2}} \cdot \frac{c^3}{a^2b^2}$.



Let OG OG' OG'' be the lines bisecting the angles A.B.C.
OR, OR', OR'' the ⊥ upon the sides.

Then the attract. of BC in direction OG' = $\frac{\mu \sin \frac{\alpha}{2}}{a}$

——— AC ——— OG = $\frac{\mu \sin \frac{\beta}{2}}{b}$

——— AB ——— OG'' = $\frac{\mu \sin \frac{\gamma}{2}}{c}$.

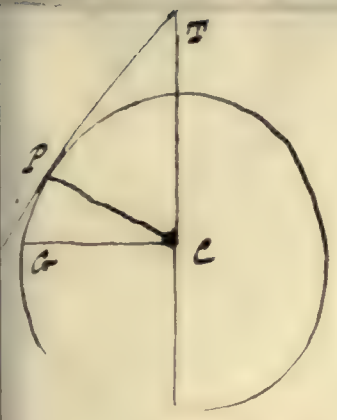
and the particle is kept at rest by these 3 forces.

∴ they are to each other as the sines of the angles contained between the others.

$$\therefore \frac{\sin \frac{\alpha}{2}}{a} : \frac{\sin \frac{\beta}{2}}{b} :: \sin\left(\frac{\gamma}{2} + \frac{\beta}{2}\right) : \sin\left(\frac{\gamma}{2} + \frac{\alpha}{2}\right)$$

$$\frac{\sin \frac{\beta}{2}}{b} : \sin \frac{\gamma}{2} : \sin\left(\frac{\gamma}{2} + \frac{\alpha}{2}\right) : \sin\left(\frac{\alpha}{2} + \frac{\beta}{2}\right).$$

$$\therefore \frac{1}{a} : \frac{1}{b} : \frac{1}{c} :: \frac{\sin\left(\frac{\gamma}{2} + \frac{\beta}{2}\right)}{\sin \frac{\alpha}{2}} : \frac{\sin\left(\frac{\gamma}{2} + \frac{\alpha}{2}\right)}{\sin \frac{\beta}{2}} : \frac{\sin\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)}{\sin \frac{\gamma}{2}}.$$



All better required proof.

PT a tangent at P.

Resolve the forces which act upon P in the direction of the tangent.

Then $\mu R = W \cos\theta$. ? where $\theta = \angle PTC$
$= \angle PCQ$.

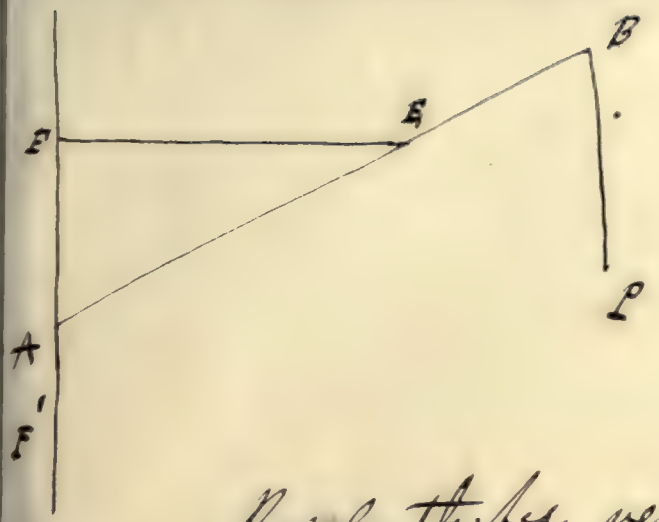
Resolve forces ⊥ tangent.

$R = W \sin\theta$.

$\therefore \mu . (W \sin\theta) = W \cos\theta$.

$\therefore \mu = \cot\theta$.

But $\mu = 1 \therefore \cot\theta = 1$ or $\theta = \frac{\pi}{4}$.



let AB = 2a. AE = x. EE = b

Resolve the fcs. vertically

$$W + L = R' \cos AEF = R' \frac{b}{x}.$$

$$\therefore R' = \frac{(W+L)x}{b} \qquad (1)$$

Take moments about A.

$$W a \cos\theta + 2La\cos\theta = R'x.$$

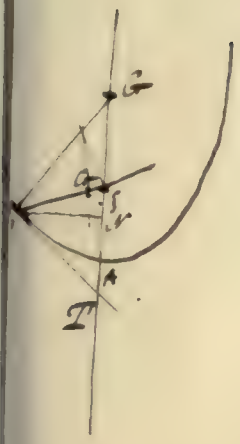
$$\frac{ab}{x^2}\{W + 2L\} = R' \qquad (2)$$

Equate (1) & (2) and reduce

$$\frac{(W+L)x}{b} = \frac{ab}{x^2}(W+2L)$$

$$x^3 = \frac{ab^2(W+2L)}{W+L}$$

$$x = \left(\frac{ab^2(W+2L)}{W+L}\right)^{\frac{1}{3}}$$



From the nature of the $\triangle$ $SA = SP = SG$

$\therefore \angle PSA = \theta = 2 \angle PGS$

Let $AT = k$. PT a tangent at P $\therefore \mu_{n+1} = \frac{2m}{4}$.

$$= \frac{2m}{PN} = \frac{2m}{PS \sin\theta} = \frac{m}{(a+k) \sin\frac{\theta}{2} \cos\frac{\theta}{2}}.$$

where $a = PG = \frac{1}{2}$ length of the rod. and $k = GT$.

$\therefore \tan PGS = \frac{1}{m}(a+k) \sin\frac{\theta}{2} \cos\frac{\theta}{2} \times$ [$m \tan\theta = \frac{2}{m} \cdot (a+k) \sin\frac{\theta}{2} \cos\frac{\theta}{2}$

$\therefore \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{2}{m} \cdot (a+k) \sin\frac{\theta}{2} \cos\frac{\theta}{2}$.

$\times$ [$\cos\frac{\theta}{2} \tan\frac{\theta}{2} = \frac{1}{m}(a+k) \sin\frac{\theta}{2}$ $\therefore m = (a+k) \cos^2\frac{\theta}{2}$. (1)

Also for the mechanical method

$$(a+k)\tan\frac{\theta}{2} = k\tan\theta = \frac{k\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$$\therefore \frac{a}{k} + 1 = \frac{2}{1-\tan^2\frac{\theta}{2}} \quad \therefore \frac{k}{a} = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

$$\therefore a+k = \frac{2a}{1+\tan^2\frac{\theta}{2}} = \frac{2a}{\sec^2\frac{\theta}{2}} = 2a\cos^2\frac{\theta}{2} \quad (2)$$

Substitute this value in (a+k) = (1)

$$\therefore m = 2a\cos^4\frac{\theta}{2}.$$

$$\therefore \cos\frac{\theta}{2} = \left(\frac{m}{2a}\right)^{\frac{1}{4}}$$

$$\text{or } \theta = 2\cos^{-1}\left(\frac{m}{2a}\right)^{\frac{1}{4}}$$

where $2a$ = length of the rod.

$$R = R' \cos\phi = \overline{T}$$

$$\left.\begin{aligned} R\sin\phi &= R'\sin\phi \\ R\cos\phi &= R'\cos\phi \end{aligned}\right\} \quad \therefore \tan\phi = \frac{\sin\phi}{\cos\phi}$$

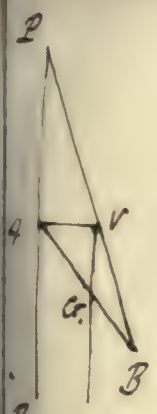
$$\overline{\tan\phi - T} = \frac{\sin\theta\cos\theta}{1-\sin^2\theta} + \frac{\sin\phi}{\cos\phi}$$

$$1 - \frac{\sin^2\theta\cos\phi}{\cos\phi(1+\sin^2\theta)}$$

$$= \sin\theta\{\cos^2\theta + 1 - \sin^2\theta\}$$

$$\cos\theta(1+\sin^2\theta - \sin^2\theta)$$

$$= \tan\theta$$



Let AB represent the lea. ϕ its inclinat. to the vertical
PB ——— string. θ its inclinat. to the vertical
$PB = b$, $AB = 2a$ —

Then $b \cos\theta = 2a \cos\phi + a \sin\phi \cot\theta$

or $b \cos\theta \sin\theta = 2a \cos\phi \sin\theta + a \sin\phi \cos\theta$ (I)

resolve forces // to PP and we have

$W - T\cos\theta = 0$. $\therefore T = \frac{W}{\cos\theta}$ (II)

Take moments about A.

$W \cdot a \sin\phi = 2a\, T \sin(\phi - \theta)$ (III)

substitute in (I) the value of T from (II) and reduce we shall have

$$\tan\phi = 2\tan\theta. \quad \text{(IV)}$$

Back from (I) $b \cos\theta \sin\theta = 2a \cos\phi \sin\theta + a \sin\phi \cos\theta$

$$b \sin\theta = 2a \cos\phi \cdot \tan\theta + a \sin\phi.$$

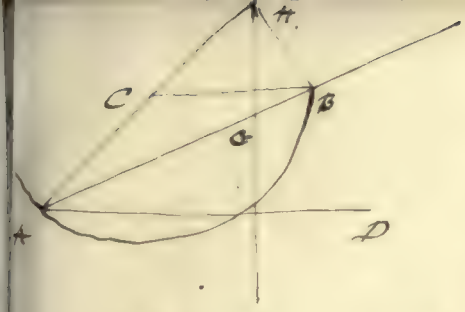
Now $\cos\phi = \frac{1}{\sqrt{1+\tan^2\phi}} = \frac{1}{\sqrt{1+4\tan^2\theta}}$ also $\sin\phi = \frac{2\tan\theta}{\sqrt{1+4\tan^2\theta}}$

$$\therefore b \cdot \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{4a\tan\theta}{\sqrt{1+4\tan^2\theta}}$$

$$\therefore 16a^2(1+\tan^2\theta) = b^2(1+4\tan^2\theta) \quad \text{or } \tan\theta = \sqrt{\frac{16a^2-b^2}{4b^2-16a^2}}$$

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \left(\frac{16a^2-b^2}{3b^2}\right)^{\frac{1}{2}}. \quad \text{also } \tan\phi = \frac{\tan\theta}{2} = \sqrt{\frac{16a^2-b^2}{16b^2-64a^2}}$$

$$\therefore \sin\phi = \left(\frac{16a^2-b^2}{12a^2}\right)^{\frac{1}{2}}.$$



See Question p. 63.

Let $AG = a$. $GB = x$. $\angle BAD = \theta$. A the point in which the direction of the forces meet. R the resistance at B. R' that at A. G the centre of gravity of the beam. C the centre of the $\odot$ AB must pass through C because the resultance at A is in the direct. of the normal.

Let r = rad of the hemi $\therefore CA = CB = r$. also $\theta = \angle BAD = \angle ABC = \angle CAB$.

Resolve forces // to AD.

$$R \sin\theta = R' \cos 2\theta \quad (I)$$

$$\text{also } r + r\cos 2\theta = (a+x)\cos\theta. \quad (II)$$

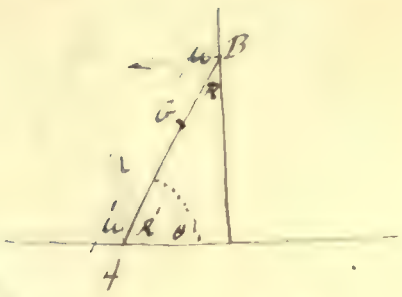
$$\text{now } x = BA \tan\theta = AB \tan^2\theta = (a+x)\tan^2\theta$$

$$\therefore x = \frac{a\tan^2\theta}{1-\tan^2\theta} \quad \text{and } (a+x) = \frac{a}{1-\tan^2\theta} = \frac{a\cos^2\theta}{\cos^2\theta - \sin^2\theta}$$

Substitute in (II) and reduce $r\{\sin^2\theta + \cos^2\theta + \cos^2\theta - \sin^2\theta\} = \dfrac{a\cos^3\theta}{2\cos^2\theta - 1}$

$$2r = \frac{a\cos\theta}{2\cos^2\theta - 1} \quad \therefore \quad 4r\cos^2\theta - a\cos\theta - 2r = 0$$

Certain forces necessary to support the beam upon the point of descending.



$R' + \mu R = W.$

Certain forces tending.

$R = \mu' R'$ $\quad \therefore R' + \mu\mu' R' = W$ or $R' = \dfrac{W}{(1+\mu\mu')}$

Take Moments about A. $\quad AC = a.\ AB = b.$

$$R\,b\sin\theta + \mu R\,b\cos\theta = a\mu'\cos\theta = a W\cos\theta$$

But $R = \dfrac{W\mu'}{1+\mu\mu'}$

$$\therefore \frac{W.\mu' b}{1+\mu\mu'}\left\{\sin\theta + \mu\cos\theta\right\} = a\mu'\cos\theta.$$

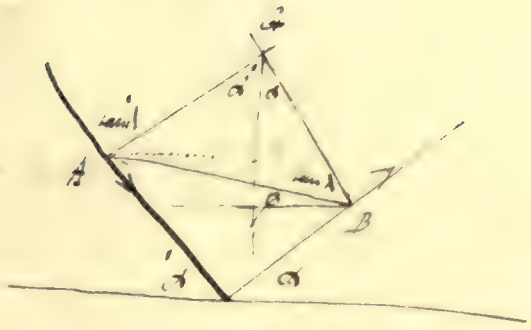
$$\frac{\mu' b}{1+\mu\mu'}\left\{\tan\theta\right. = a - \frac{\mu\mu' b}{1+\mu\mu'}$$

$$= \frac{a(1+\mu\mu') - \mu\mu' b}{1+\mu\mu'}$$

$$\therefore \tan\theta = \frac{a(1+\mu\mu') - \mu\mu' b}{\mu' b}$$

or if θ' = Angle of Repose from horizon when a is

a maximum $\quad \tan\theta = \dfrac{a - \mu\mu' b}{(a+b)\mu'}$



Suppose B on the point of sliding

Certain forces = Zero.

$$R\left(\sin\alpha \mp \tan\lambda \cos\alpha\right) = R'\left(\sin\alpha' + \tan\lambda' \cos\alpha'\right)$$

$$\therefore R = \frac{R'\cos\lambda'}{\cos\lambda}\,\frac{\sin\alpha'\cos\lambda' + \sin\lambda'\cos\alpha'}{\sin\alpha\cos\lambda - \sin\lambda\cos\alpha}.$$

$$\therefore \frac{R}{\cos\lambda} = R'\,\frac{\cos\lambda'}{\cos\lambda}\left\{\frac{\sin(\alpha'+\lambda')}{\sin(\alpha-\lambda)}\right.$$

Resolve forces vertically.

$$R'\left(\cos\alpha' - \tan\lambda'\sin\alpha'\right) + R\left(\cos\alpha + \tan\lambda\sin\lambda\right) = W.$$

$$\frac{R'}{\cos\lambda'}\left\{\cos(\alpha'+\lambda')\right. + \frac{R}{\cos\lambda}\cos\cos(\alpha-\lambda) = W.$$

$$\therefore \frac{R'}{\cos\lambda'}\left\{\cos(\alpha'+\lambda')\right. + \frac{\sin(\alpha'+\lambda')\cos(\alpha-\lambda)}{\sin(\alpha-\lambda)} = W.$$

$$\text{or } \frac{R'}{\cos\lambda'\sin(\alpha-\lambda)}\left\{\sin(\alpha'+\lambda'+\alpha-\lambda)\right. = W.$$

$$\therefore R' = \frac{W\cos\lambda'\sin(\alpha-\lambda)}{\sin(\alpha'+\lambda'+\alpha-\lambda)}$$

$$\therefore R = \frac{W\cos\lambda.\sin(\alpha'+\lambda')}{\sin(\alpha'+\lambda+\alpha-\lambda)}.$$

Also taking moments about P.

$$\frac{W\alpha}{2}\cos\zeta + \ldots = R'\left(a\sin(\xi-x') - a\cos(\xi-x')\right.$$

$$\frac{\cos\zeta}{2} = \frac{\cos\lambda'\sin(\alpha-\lambda)}{\sin(\alpha'+\lambda'+\alpha-\lambda)}\left\{\cos\xi\cos\alpha' + \sin\xi\sin\alpha' - \tan\lambda'(\sin\xi\cos\alpha' - \cos\xi\sin\alpha'\right.$$

$$\frac{\sin(\alpha'+\lambda'+\alpha-\lambda)}{2\cos\lambda'\sin(\alpha-\lambda)} = \cos\alpha' + \tan\zeta\sin\alpha' - \sin\lambda'\left(\frac{1}{\cos\xi}+\tan\zeta\tan\xi\right)$$

$$= \frac{\cos\alpha'\cos\lambda' + \sin\lambda'\sin\alpha'}{\cos\lambda'} + \tan\zeta\left\{\frac{\sin\alpha'\cos\lambda - \cos\alpha'\sin\lambda}{\cos\lambda'}\right.$$

$$\frac{\sin(\alpha'+\lambda'+\alpha-\lambda) - 2\cos(\alpha'+\lambda')\sin(\alpha-\lambda)}{2\cos\lambda'\sin(\alpha-\lambda)} = \frac{1}{2}\tan\zeta\cdot\frac{\sin(\alpha'+\lambda')}{\cos\lambda'}$$

$$\text{or } \frac{1}{2}\tan\zeta = \frac{\sin(\alpha'+\lambda')\cos(\alpha-\lambda) - \cos(\alpha'+\lambda')\sin(\alpha-\lambda)}{\sin(\alpha'+\lambda')\sin(\alpha-\lambda)}$$

$$= \cot(\alpha-\lambda) - \cot(\alpha'+\lambda')$$

Resolve forces // to the plank.

$\mu R = (W + A) \sin\theta$.

Resolve for $\perp$ to the plank.

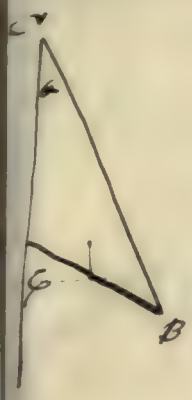
$R = (W + A) \cos\theta$.

$\therefore \mu = \tan\theta = \text{hard} \therefore \theta = 1$

Take moments about B.

$A(a - r\theta) = W . r\theta$

$\therefore \frac{A}{W} = \frac{r\theta}{a - r\theta}$.



$AB = a$. $CB = l$.

Resolve for. vertically.

$$\mu R + W = T \cos\theta \quad (1)$$

$$R = T \sin\theta \quad (2)$$

Take moments about A.

$$T\, a \sin(\phi - \theta) = W \cdot \frac{a}{2} \sin\phi$$

$$2T \sin(\phi - \theta) = W \sin\phi. \quad (3)$$

From the geometry

$$a \sin\phi = l \sin\theta \quad (4)$$

From (2) we have $T = \frac{R}{\sin\theta}$, which substituted gives

$$\mu R + W = R \cos\theta$$

$$2R \sin(\phi - \theta) = W \sin\phi \sin\theta$$

$$R = \frac{W \tan\phi \tan\theta}{2(\tan\phi - \tan\theta)} \quad (5)$$

But we have seen that $\mu R + W = R \cos\theta$

$$\therefore R = \frac{W}{\cot\theta - \mu} \quad (6)$$

Equate (5) & (6) and we have.

$$\frac{\tan\phi \tan\theta}{\tan\phi - \tan\theta} = \frac{2}{\cot\theta - \mu}. \quad \text{or} \quad \tan\phi = \frac{2\tan\theta}{1 + \mu\tan\theta}.$$

$$\tan\varphi = \frac{2\tan\theta}{1+\mu\tan\theta} \quad \therefore \tan^2\varphi = \frac{4\tan^2\theta}{1+2\mu\tan\theta+\mu^2\tan^2\theta}$$

$$\frac{\sin^2\varphi}{1-\sin^2\varphi} = \frac{4\tan^2\theta}{1+2\mu\tan\theta+\mu^2\tan^2\theta}$$

$$\left(1+2\mu\tan\theta+(\mu^2+4)\tan^2\theta\right)\sin^2\varphi = 4\tan^2\theta$$

$$\therefore \sin^2\varphi = \frac{4\tan^2\theta}{1+2\mu\tan\theta+(\mu^2+4)\tan^2\theta}.$$

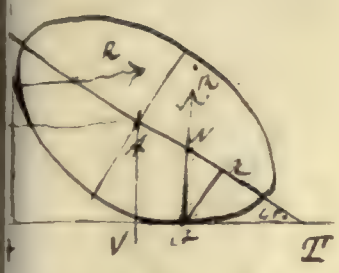
But from (4) $\sin^2\varphi = \frac{l^2}{a^2}\sin^2\theta$

$$\therefore \frac{l^2}{a^2}\sin^2\theta = \frac{4\frac{\sin^2\theta}{\cos^2\theta}}{1+2\mu\tan\theta+(\mu^2+4)\tan^2\theta}$$

$$\frac{l^2}{a^2} = \frac{4\sec^2\theta}{1+2\mu\tan\theta+(\mu^2+4)\tan^2\theta}$$

$$= \frac{4(\tan^2\theta+1)}{1+2\mu\tan\theta+(\mu^2+4)\tan^2\theta}$$

$$\therefore \left(4a^2-4l^2-\mu^2l^2\right)\tan^2\theta - 2\mu l^2\tan\theta + 4a^2 - l^2 = 0$$



At Q $\quad y = \frac{b}{a}\sqrt{a^2 - x^2}$.

$\therefore \frac{dy}{dx} = -\frac{b}{a} \cdot \frac{x}{\sqrt{a^2 - x^2}} = 1$

$\therefore \frac{b^2}{a^2} \frac{x^2}{a^2 - x^2} = 1$

$b^2 x^2 = a^4 - a^2 x^2$

$x^2 = \frac{a^4}{a^2 + b^2} \quad x = \frac{a^2}{\sqrt{a^2 + b^2}}$

$\therefore$ At R $= x = \frac{a^2}{\sqrt{a^2 + b^2}}$

$y = \frac{b}{a}\sqrt{a^2 - x^2} = \frac{b}{a}\sqrt{a^2 - \frac{a^4}{a^2 + b^2}} = \frac{b}{a}\sqrt{\frac{ab}{a^2 + b^2}} = \frac{b^2}{\sqrt{a^2 + b^2}}$

$\therefore$ Q G $=$ A N $= \frac{b^2}{\sqrt{a^2 + b^2}} \quad \therefore$ A N $= \frac{a^2 - b^2}{\sqrt{a^2 + b^2}} = \frac{x l^2}{\sqrt{a^2 + b^2}}$

Now for $=^{m}$ plane $\quad p = W$.

$\mu R' = R \quad \therefore R = \mu W$.

Also $W \times Q V = R \times A x' = \mu W \times A x'$

for $\Delta^{le} Q V = A N \times 45^{\circ} = \frac{a^2 l^2}{\sqrt{a^2 + b^2}} \cdot \frac{1}{\sqrt{2}}$.

$A x' = A G = A V + V G$.

$= \frac{A T}{\sqrt{2}} + \frac{a^2 l^2}{\sqrt{a^2 + b^2}} \frac{1}{\sqrt{2}}$

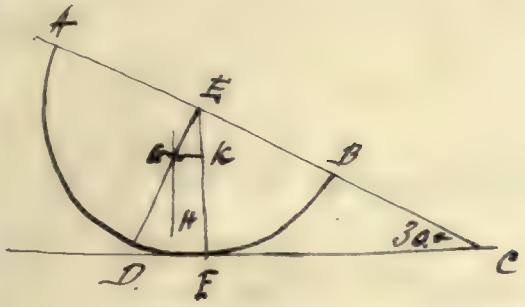
$A T = \frac{a^2}{x} = \frac{a^2}{\frac{a^2}{\sqrt{a^2 + b^2}}} = \sqrt{a^2 + b^2}$

$\therefore A G = \frac{1}{\sqrt{2}} \left\{ \sqrt{a^2 + b^2} + \frac{x l^2}{\sqrt{a^2 + b^2}} \right\} = \frac{1}{\sqrt{2}} \left\{ \frac{a^2 + b^2 + a^2 l^2}{\sqrt{a^2 + b^2}} \right\}$.

$\therefore \frac{W}{\sqrt{2}} \cdot \frac{a^2 l^2}{\sqrt{a^2 + b^2}} = \mu \frac{W}{\sqrt{2}} \left\{ \frac{a^2 + b^2 + a^2 l^2}{\sqrt{a^2 + b^2}} \right\} = \frac{\mu W}{\sqrt{2}} \left\{ \frac{2a^2}{\sqrt{a^2 + b^2}} \right\}$

$a l^2 = \mu 2 a^2$.

$\therefore \mu = \frac{1}{2} l^2$.



To find c. of g. of the hemisphere.

$$X = \frac{\int x\, dy^2}{\int dy^2} = \frac{\int x\, dx^2 - x^3}{\int a^2 - x^2} = \frac{\frac{a^4}{4}}{\frac{2}{3}a^3} \quad \text{Let } r = 0.1 = a$$

$$= \frac{3}{8}a. = EG. \quad (1)$$

Draw EE vertical.

The weight acts vertically downwards = GH.

and the moment of G about EE = G × GK

$$= G.\ GE \sin 30°.$$

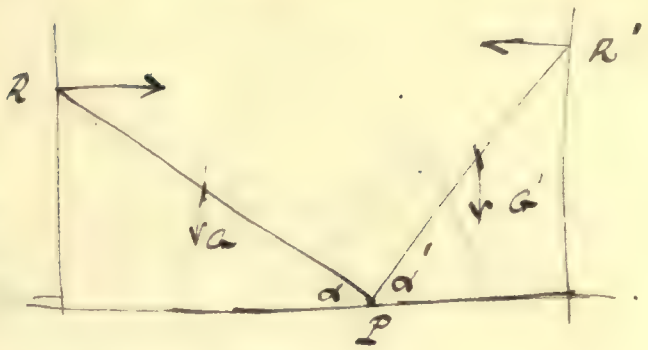
$$= G.\ \frac{GE}{2}.$$

where G = weight of the ½ sphere.

But $GE = \frac{3}{8}a.$ from (1)

$\therefore$ the moment of G = $G.\ \frac{3}{16}a$

and a couple whose moment is equal to this is necessary to preserve = m.



Let P be the mutual pressure of P.
This is exerted horizontally.
Then for the $\perp^{m}$ of RP we have.

$$R = P \qquad (1)$$

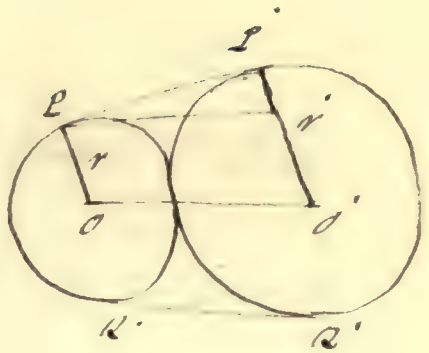
$$P G \cos\alpha W = P R \sin\alpha . R$$

$$= 2 P G \sin\alpha P .$$

$$\therefore P = \frac{W . \cos\alpha}{2} .$$

$$\text{Similarly } P = \frac{W' \cos\alpha'}{2} .$$

$$\therefore W \cos\alpha = W' \cos\alpha'$$



PP', RR', vertical sections of the cylinders PR, $R'R'$ the shortest distance of centres of the Cyl.

PP' the point about the thing cause the cylinders.

OP is endwise to $O'P'$. Let Pr' be $\perp$ to OO'.

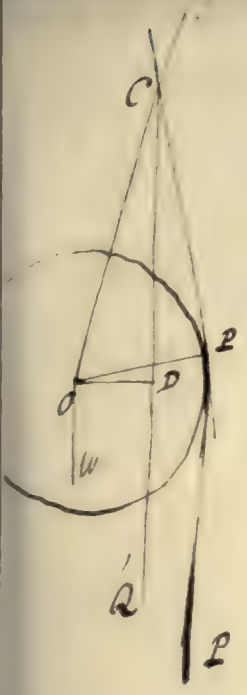
Then in $\dot{P}P'O$ $= \frac{P'r'}{Pr'} = \frac{r'-r}{r'+r}$ $\therefore$ Cot $\dot{P}PO = \frac{2\sqrt{rr'}}{r'+r}$

with F = mutual press. of the cyl.rs. T = tension of thing

Then $2T$ Cos $\dot{P}PO = F$.

or $\frac{4\sqrt{rr'}}{r'+r} T = F$.

$\therefore \frac{F}{T} = \frac{4\sqrt{rr'}}{r+r'}$



C the point of suspension CD a vertical line thro' C. $\angle OCP = \theta$

$\angle OCD = \theta'$.

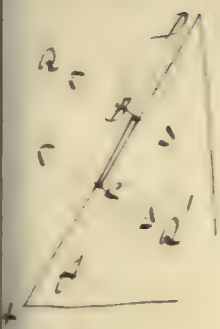
The centre of gravity of the whole mass must lie in the vertical line CQ', and the whole pressure downwards is equal to a pressure $P+W\,(=R)$ in CQ

∴ Taking moments about O.

$$R \times OD = P \times OP.$$

$$R \times b \sin\theta' = aP.$$

$$\sin\theta' = \frac{aP}{bR}.$$



Taking moments round C, of R & CD. and of R' the tension of the string at B.

Then $CB \times R = CD \times W \cos\theta$.

$$\text{or } R = \frac{CD}{CB} \cdot W \cos\theta$$

This acts in direction BC.

Taking moments about B and of R' the tension of the string at C.

Then $BC \times R' = BD \times W \cos\theta$.

$$\therefore R' = \frac{BD}{BC} \cdot W \cos\theta.$$

This acts in direction CR'

Also the pressure on the length of $AB = W \sin\theta$.

CB. BB'. B'C' the rad. of the Curl. of the O. BC vertical

Then $\tan AB' = \frac{1}{2}$. $\tan CB'A = \tan CB'J = 2$.

$\therefore \tan CB'C' = \frac{2-\frac{1}{2}}{1+1} = \frac{3}{4}$

$\therefore \sin CB'C = \frac{3}{5} = \sin C$.

Then for $= ^{m}$ of BC' where R = pressure at C

Then. $2R = \frac{2}{2} \cdot W \sin C = \frac{3W}{10}$ w. where w = weight BC'

$\therefore R = \frac{3}{40} W'$ of W' $= 4w$.

Let T be the vertical thrust upon the hinge B'

$W = T + R \sin C = T + \frac{3w}{10} \times \frac{3}{5}$

$\therefore T = \frac{7}{10} w = \frac{3.28}{5.10}$ of W' $=$

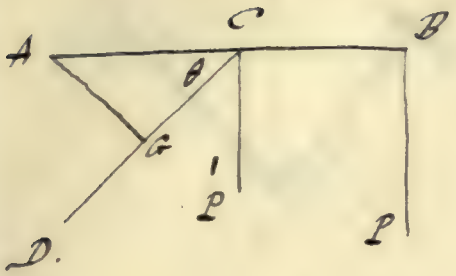
Therefore if R'' vertical pressure at A

$\therefore R' = \frac{w'}{2} + \frac{2.3.28\ w}{5.10}$

$\therefore T = w - \frac{9w}{50} = \frac{41w}{50} = \frac{41 w'}{200}$

Therefore if R' = vertical pressure at A

$R' = \frac{w'}{2} + \frac{2.41.w'}{200} = \frac{182\ w}{200} = \frac{91\ w}{100}$.



Let P' be a weight such that being applied at C it shall produce the same effect as P does applied at B. Then $P' \times AC = P.AB$ or $P' = \frac{AB}{AC}.P.$

Let S = Resistance on the Inclined Plane CD.

$\therefore AC \times P' = AG \times S = AC \sin\theta . S.$

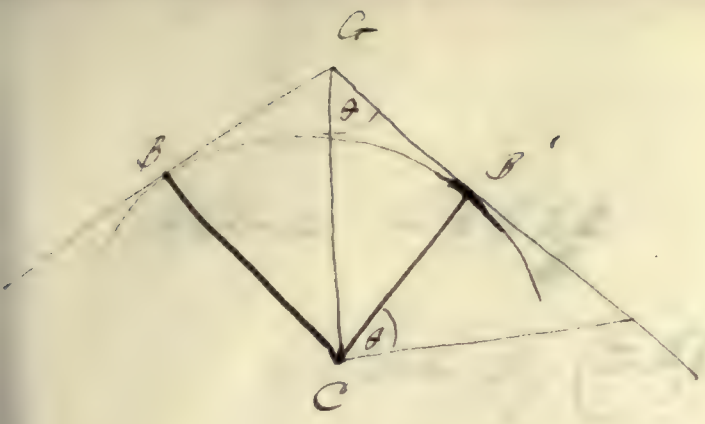
$$\therefore S = \frac{P'}{\sin\theta}.$$

Let X. Y be the horizontal and vertical pressures upon the Inclined Plane

Then $X = S\cos\theta = P' \tan\theta = P'\frac{AC}{AD} = P.\frac{AB}{AD}.$

$Y = S\sin\theta = P' \qquad\qquad = P.\frac{AB}{AC}.$

and the whole pressure $= S = \frac{P'}{\sin\theta} = P.\frac{AB}{AC}\left\{\frac{AD^2}{AD^2+AC^2}\right\}^{\frac{1}{2}}.$



C the centre of the ⊙

R = reaction at B.

Take moments about G.

$R \cdot r\cot\theta = W a \sin\theta$ if $2a$ = length of the beam.

$\therefore R = \dfrac{W \cdot a \sin^2\theta}{r\cos\theta}$.

Now $W = R\sin\theta = \dfrac{W a \sin^3\theta}{r\cos\theta}$

$\therefore r\cot\theta = a\sin^3\theta$.

Assume $m = \beta \frac{dx}{\varepsilon}$

whence $d_s\left(t \frac{dx}{s}\right) = 0, \quad d_s\left(t \frac{dy}{s}\right) = g\beta \frac{dx}{\varepsilon}$.

Now $d_s\left(t \frac{dx}{s}\right) = 0 \quad \therefore t \frac{dx}{s} = C$.

Let τ = tension of the thing when $x = 0$

$$\therefore t \frac{dx}{s} = \tau.$$

also $d_s\left(t \frac{dy}{s}\right) = d_s\left(t \frac{dx}{s} \cdot \frac{dy}{x}\right) = d_s\left(\tau \frac{dy}{x}\right)$

wherefore $d_s\left(\tau \frac{dy}{x}\right) = g\beta \frac{dx}{\varepsilon}$.

Integrate $\tau \frac{dy}{x} = g\beta x + C$.

$\frac{dy}{x} = 0$ when $x = 0 \quad \therefore C = 0$

Integrate again and whence

$$\tau y = \frac{1}{2} g\beta x^2 + C'$$

$y = 0$ when $x = 0 \quad \therefore C' = 0$

$$\text{wherefore } x^2 = \frac{2\tau}{g\beta} y$$

Assume $m = \beta x . \frac{dx}{ds}$.

Then the $=^{ns}$ are.

$$\frac{d}{ds}\left(\pm \frac{dx}{ds}\right) = 0 \quad (1) \qquad \frac{d}{ds}\left(\pm \frac{dy}{ds}\right) = \beta x \frac{dx}{ds}$$

From (1) we have $\pm \frac{dx}{ds} = C$ a const.

Let τ = tension at the chain at its lowest point $\therefore$ when $x = 0$

$$C = \tau$$

$$\therefore \pm \frac{dx}{ds} = \tau \quad (3)$$

From (2) $\frac{d}{ds}\left(\pm \frac{dy}{ds}\right) = \left\{\frac{d}{ds}\left(\tau \frac{dx}{dy}\right) \text{ from (3)}\right\} = \beta x . \frac{dx}{ds}$

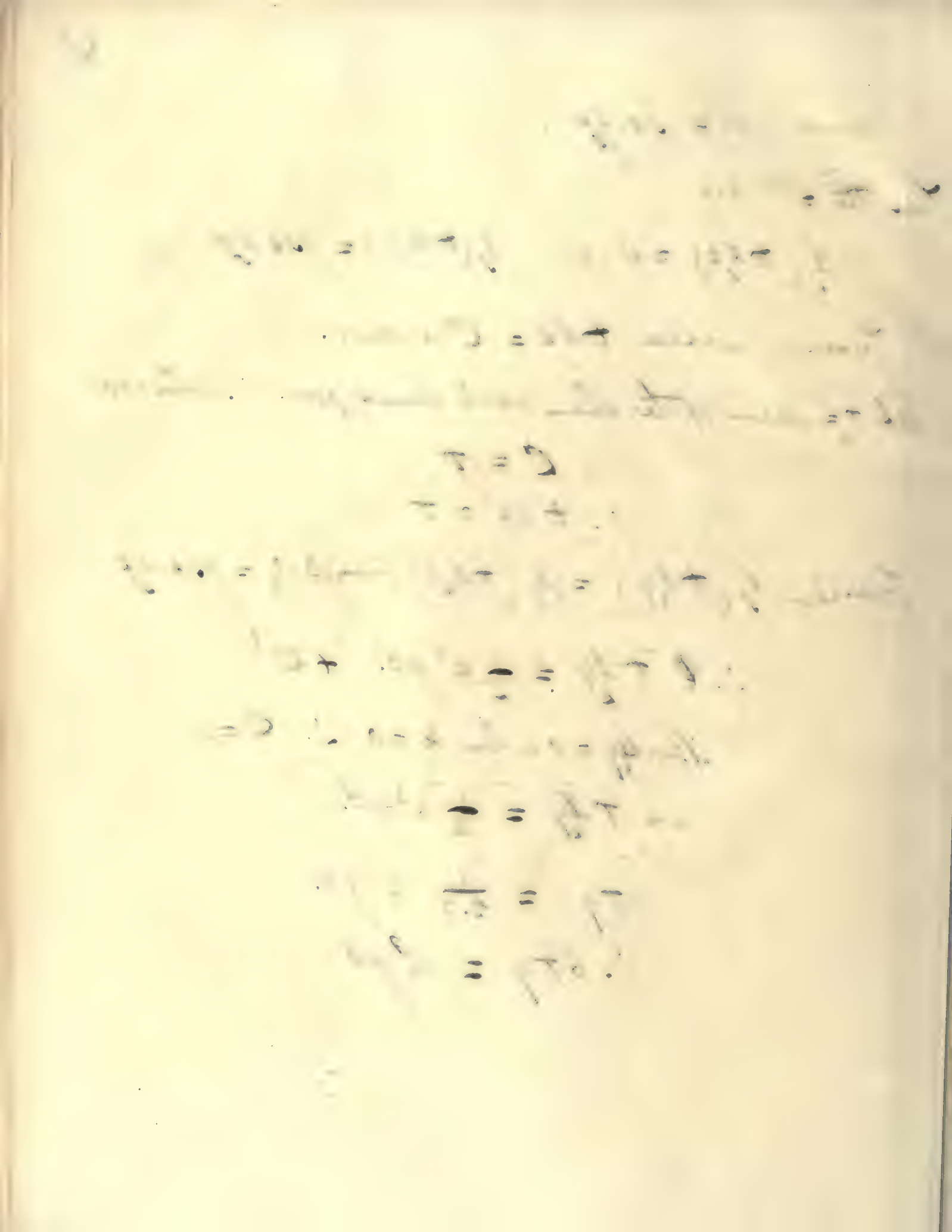
$$\therefore \tau \frac{dy}{dx} = \frac{1}{2} x^2 \beta . + C'$$

But $\frac{dy}{dx} = 0$ when $x = 0$ $\therefore$ $C = 0$.

$$\text{and } \tau \frac{dy}{dx} = \frac{1}{2} x^2 \beta .$$

$$\tau y = \frac{1}{2.3} x^3 \beta .$$

$$\therefore 6 \tau y = x^3 \beta$$



Assume $w = \beta x^{\frac{1}{2}} d_s x$.

The $=^{n}$ are $d_s(\pm d_s x) = 0$ $d_s(\pm d_s y) = g\beta x^{\frac{1}{2}} d_s x$

$d_s(\pm d_s x) = 0 \therefore \pm d_s x = C$.

Let τ = least of the chain which bound points on which $x = 0$ $\therefore \pm d_s x = \tau$

also $d_s(\pm d_s y) = d_s(\pm d_s x\, d_x y) = d_s(\tau d_x y)$

Therefore $d_s(\tau d_x y) = g\beta x^{\frac{1}{2}} d_s x$

Integrate and reduce.

$$\tau d_x y = \frac{2}{3} g\beta x^{\frac{3}{2}} + C.$$

$d_x y = 0$ when $x = 0$ $\therefore C = 0$.

$$\tau y = \frac{2}{3} \cdot \frac{2}{5} \cdot g\beta x^{\frac{5}{2}}.$$

$$\text{or } (15)^2 \tau^2 y^2 = 4^2 \cdot g^2 \beta^2 x^5$$

here minus.

$$\int_s (\pm dx) = 0 \quad , \quad \int_s (\pm dy) = +mg. \qquad (2)$$

$\therefore \pm \int_s dx = C.$ $\therefore T =$ tension at the lowest point of the

string minus $T = C$

$$\therefore \pm \int_s dx = T.$$

from $\pm \int_s dy = \pm \int_s dx \int_s dy = T \int_x dy.$

Hence (2) becomes. $\int_s \left(T \int_x dy \right) = +mg.$

$$\therefore T \int_x dy = g \int_s m.$$

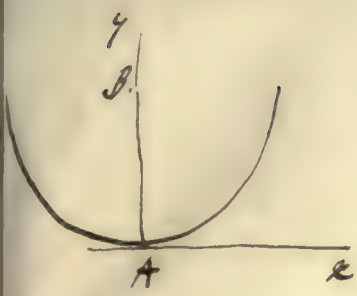
In the case before us $m = \beta y^n$ sin $\phi = \beta y^n \cdot \int_s dy$

$$\therefore T \int_x dy = g \int_s \beta y^n \int_s dy = \frac{\beta g\, y^{n+1}}{n+1}$$ where β is const.

$$\therefore \int_y dx = \frac{(n+1)\, T}{\beta g \cdot y^{n+1}}$$

$$x = -\frac{(n+1)}{n} \cdot \frac{T}{\beta g \cdot y^n}$$

or $x \propto y^n = -\frac{(n+1)}{n} \cdot \frac{T}{\beta g}$ the required $= n.$



Taking the lowest point of the D for the origin the eqn is $x^2 = 4ay$ or $y = \frac{x^2}{4a}$.

If a' = latus rectum eqn is $y = \frac{x^2}{a'}$

Now $m = \frac{T}{g} \frac{d^2y}{ds}_x$.

$y = \frac{x^2}{a'} \therefore \frac{dy}{dx} = \frac{2x}{a'}$ $\frac{d^2y}{dx^2} = \frac{2}{a'}$

also $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{4x^2}{a'^2}} = \frac{\sqrt{a'^2+4x^2}}{a'}$

$\therefore m = \frac{T}{g} \frac{\frac{2}{a'}}{\frac{\sqrt{a'^2+4x^2}}{a'}} = \frac{T}{g} \cdot \frac{2}{\sqrt{a'^2+4x^2}}$

To find absolut point this is a ~~maximum~~ minimum

$m = \frac{T}{g} \cdot \frac{2}{\sqrt{a'^2+4x^2}} \therefore \frac{dm}{dx} = -\frac{T}{g} \cdot \frac{\delta x}{\left(a'^2+4x^2\right)^{\frac{3}{2}}} = 0$ if $x = 0$.

$$\frac{d^2m}{dx^2} = \frac{T}{g} \frac{\left(a'^2+4x^2\right)^{\frac{3}{2}} \cdot \delta - \delta x \cdot \frac{3}{2}\left(a'^2+4x^2\right)^{\frac{1}{2}} \cdot \delta x}{\left(a'^2+4x^2\right)^{\frac{3}{2}}}$$

$$= \frac{T}{g} \cdot \frac{\left(\delta a'^2 + 32x^2 - 96x^2\right)\delta}{\left(a'^2+4x^2\right)^{\frac{5}{2}}}$$ which is +ve if $x = 0$

$\therefore$ m is a minimum at the origin

$$m = \frac{d^2y}{ds}$$

$$x = a(\theta + \sin\theta) \quad , \quad y = a(1-\cos\theta).$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\sin\theta}{1+\cos\theta}\right) = \frac{d}{d\theta}\left(\frac{\sin\theta}{1+\cos\theta}\right)\frac{d\theta}{dx}.$$

$$\frac{d}{d\theta}\left(\frac{\sin\theta}{1+\cos\theta}\right) = \frac{(1+\cos\theta)\cos\theta + \sin^2\theta}{(1+\cos\theta)^2} = \frac{1}{1+\cos\theta}$$

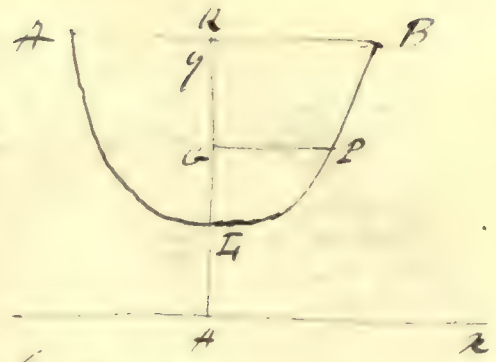
$$\therefore \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{\sin\theta}{1+\cos\theta}\right)\frac{d\theta}{dx} = \frac{1}{1+\cos\theta} \times \frac{1}{a(1+\cos\theta)} = \frac{1}{a(1+\cos\theta)^2}$$

$$\text{also } \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{\sin^2\theta}{(1+\cos\theta)^2}} = \frac{\sqrt{2}}{\sqrt{1+\cos\theta}} = \frac{1}{\cos\frac{\theta}{2}} = \sec\frac{\theta}{2}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{1}{a(1+\cos\theta)^2} = \frac{1}{4a\cos^4\frac{\theta}{2}} = \frac{1}{4a}\sec^4\frac{\theta}{2}.$$

$$\therefore m = \frac{T}{g} \cdot \frac{\sec^4\frac{\theta}{2}}{4a\sec\frac{\theta}{2}} = \frac{T}{g} \cdot \frac{\sec^3\frac{\theta}{2}}{4a}$$

$$\frac{1}{\rho} = T\frac{d^2s}{dx^2} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}.$$



Let I be the lowest point in the curve. c the length of chain whose length = tension at I. $AI = c$

y being measured from A from the point.

Then the eqn to the curve considering A as the origin is

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right).$$

∴ tension at any point P = gmy of m = unit of mass.

$$= \frac{gm}{2} \cdot c\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right).$$

wherefore if $CB = CA = a$. the tension at $B = \frac{1}{2}gmc\left\{e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right\} = T$

and this must be a minimum.

$$\therefore \frac{1}{c}T = \frac{1}{2}gm\left\{e^{\frac{a}{c}} - e^{-\frac{a}{c}} + \frac{a}{c}\left(-e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right)\right\} = 0.$$

$$\text{or } e^{\frac{2a}{c}}\left(1 - \frac{a}{c}\right) = \frac{a}{c} + 1$$

$$\therefore e^{\frac{a}{c}} = \left(\frac{\frac{a}{c} + 1}{\frac{a}{c} - 1}\right)^{\frac{1}{2}} \quad (1)$$

Then $l = \frac{c}{2}\left(e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right)$ where l = length of IB.

∴ whole length of the catenary is $c\left(e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right)$ (2).

The value of c will be determined from (1) and substituted in (2) for any case.

The ^{n} Latte curve is $y = \frac{x^4}{a^3}$

$\therefore \frac{dy}{dx} = \frac{4x^3}{a^3}$. $\frac{d^2y}{dx^2} = \frac{3.4x^2}{a^3}$, $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$= \frac{\sqrt{a^6+16x^6}}{a^3}$.

Now $m = \frac{r}{g} \cdot \frac{\frac{d^2y}{dx^2}}{\frac{ds}{dx}} = \frac{3.4.x^2}{\sqrt{a^6+16x^6}}$.

and this must be a maximum

Now $m = \frac{3.4.x^2}{\sqrt{a^6+16x^6}}$ (1)

$\frac{dm}{dx} = \frac{\sqrt{a^6+16x^6}.24x - \frac{12x^2.96}{2\sqrt{a^6+16x^6}}}{a^6+16x^6}$

$= \frac{24a^6x - 192x^7}{(a^6+16x^6)^{\frac{3}{2}}} = 0$ if $x = 0$ or $x = \frac{a}{\sqrt{2}}$.

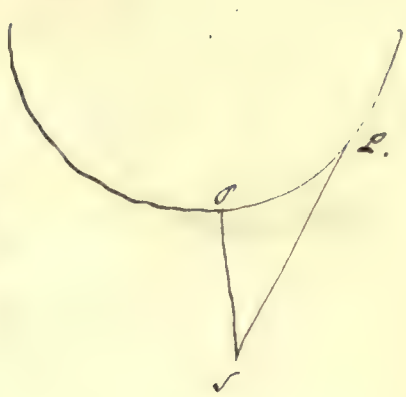
the former of these values gives a minimum, the latter a maximum value

substituting the latter value of x in (1) we have

$m = \frac{3.4.\frac{a^2}{2}}{a^3\sqrt{3}} = \frac{2\sqrt{3}}{a}$

also $y = \frac{x^4}{a^3} = \frac{a^4}{4a^3} = \frac{a}{4}$.

O the lowest point of the catenary.
S the centre of force. P a point
in the catenary. PS = r . SO = c.



Then if T = tension at O resolved
from the equation $T = \frac{m c^{\mu+1}}{\mu+1}$.

Let T = tension at P.

Then $T = m \int \frac{r^\mu}{c^\mu} \cdot c^\mu = \frac{m \cdot r^{\mu+1}}{\mu+1} + C$.

when $T = T$. $r = c$

$$\therefore T = \frac{m \cdot c^{\mu+1}}{\mu+1} + C$$

$$\text{But } T = \frac{m c^{\mu+1}}{\mu+1} \quad \therefore C = 0.$$

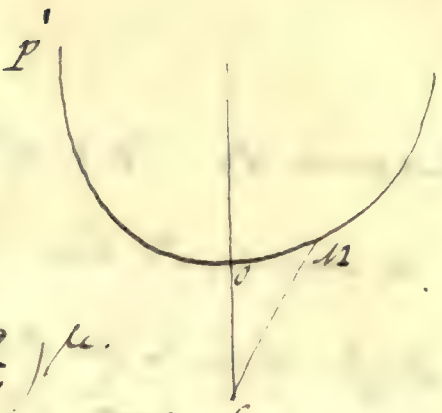
$$T = \frac{m r^{\mu+1}}{\mu+1}.$$

$$\frac{d\theta}{r} = \frac{C'}{r\sqrt{T^2 r^2 - C'^2}} = \frac{C T}{r\sqrt{T^2 r^2 + C T}} = \frac{c^{\mu+2}}{r\sqrt{r^{2\mu+4} - c^{2\mu+4}}}$$

$$= \frac{c \cdot \left(\frac{c}{r}\right)^{\mu+1}}{r^2 \sqrt{1 - \left(\frac{c^2}{r^2}\right)^{\mu+2}}} = \frac{a_p \cdot \left(\frac{c}{\mu}\right)^{\mu+2}}{(\mu+2)\sqrt{1 - \left(\frac{c^2}{\mu^2}\right)^{\mu+2}}}$$

$$\therefore \theta = \frac{1}{\mu+2} \cos^{-1}\left(\frac{c}{\mu}\right)^{\mu+2}.$$

$$\text{or } \cos\left(\frac{c}{\mu}\right)^{\mu+2} = \cos(\mu+2)\theta.$$



let P'P be the Catenary. S the centre of force, O the lowest point of the curve. OS = c. SM = r. If the force of attr. at O. Then the force of attr at M will be $k.\left(\frac{c}{r}\right)^{\mu}$.

$$\therefore R = \int F_M = k_M \cdot \frac{c^{\mu}}{r^{\mu}} = -\frac{k_M}{\mu-1} \cdot \frac{c^{\mu}}{r^{\mu-1}} + C.$$

let T be the tension at O. Then $T = -\frac{k_M c}{\mu-1} + C'$

Back from the question $T = -\frac{k_M c}{\mu-1}$ $\therefore C' = 0$.

$$\therefore R = \frac{k_M}{\mu-1} \cdot \frac{c^{\mu}}{r^{\mu-1}} = \frac{k_M c}{\mu-1} \cdot \left(\frac{c}{r}\right)^{\mu-1}$$

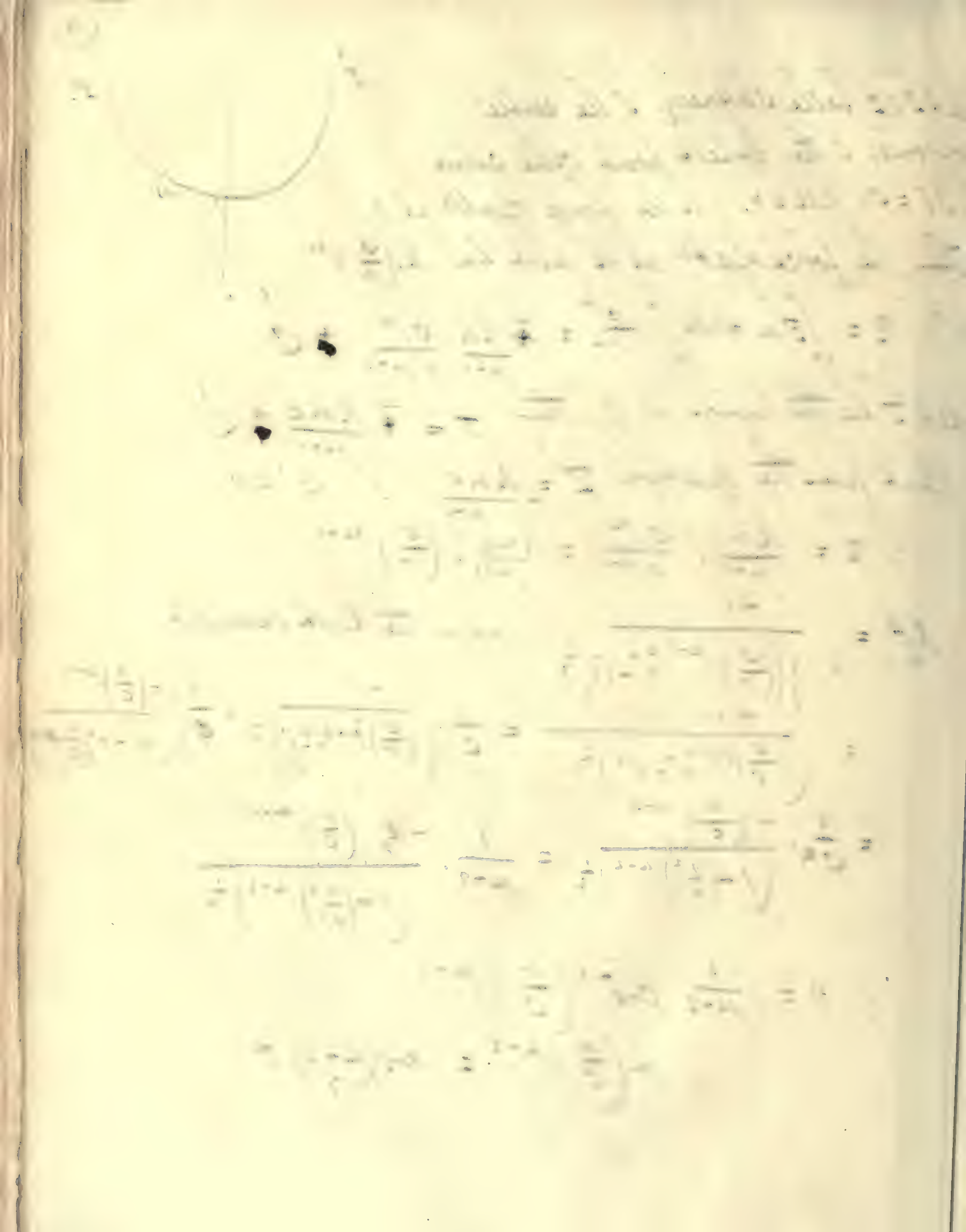
$$\frac{d\theta}{r} = \frac{-1}{r\left\{\left(\left(\frac{c}{r}\right)^{\mu-1}\right)^2 r^2 - 1\right\}^{\frac{1}{2}}} \quad \text{as in the last example.}$$

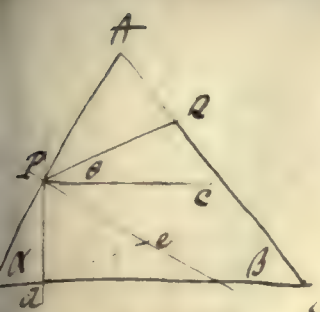
$$= \frac{-1}{\left(\left(\frac{c}{r}\right)^{2\mu-2} r^4 - r^2\right)^{\frac{1}{2}}} = \frac{1}{c} \cdot \frac{-1}{\left(\left(\frac{c}{r}\right)^{2\mu-4} r^2 - \frac{r^2}{c^2}\right)^{\frac{1}{2}}} = \frac{1}{c} \left(\frac{-\left(\frac{r}{c}\right)^{\mu-1}}{r^2 - r^2\left(\frac{r}{c}\right)^{2\mu-2}}\right)$$

$$= \frac{1}{c k_M} \cdot \frac{-\left(\frac{r}{c}\right)^{\mu-3}}{\left(1 - \left(\frac{r^2}{c^2}\right)^{\mu-2}\right)^{\frac{1}{2}}} = \frac{1}{\mu-2} \cdot \frac{-d_r \cdot \left(\frac{r}{c}\right)^{\mu-2}}{\left(1 - \left(\frac{r^2}{c^2}\right)^{\mu-1}\right)^{\frac{1}{2}}}.$$

$$\therefore \theta = \frac{1}{\mu-2} \cos^{-1}\left(\frac{r}{c}\right)^{\mu-2}$$

$$\text{or } \left(\frac{r}{c}\right)^{\mu-2} = \cos(\mu-2)\theta.$$





L. L. the positions of P and R. Pc horizontal Pd vertical & // to PR. Let T = tension of PR. Then P is kept at rest by 3 forces. R (the resistance of the plane) acting ⊥ to PQ; its weight acting ⊥ Pd, and T acting in PR in the direction of dc.

$$\therefore \frac{T}{P} = \frac{dc}{Pd} = \frac{\sin\alpha}{\cos(\alpha-\theta)} \quad \therefore T = \frac{P\sin\alpha}{\cos(\alpha-\theta)} \quad (1)$$

Similarly it may be shown that $T = \frac{R\sin\beta}{\cos(\beta+\theta)}$ (2)

From these values of T we have

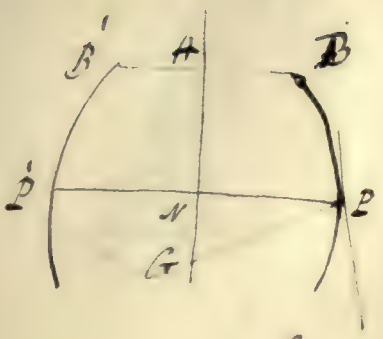
$$\tan\theta = \frac{P\cot\beta - R\cot\alpha}{P+R} \quad (3).$$

Also from Hooke's principle

$$a' = a(1+\lambda T) = a\left(1+\frac{P\lambda\sin\alpha}{\cos(\alpha-\theta)}\right) \text{ from (1).}$$

where a' = extended length of the string, and a = its natural length.

(5)



Let BP, $B'P'$ be two branches of the required curve symmetrically situated with respect to AN which is vertical, and bisects $B'B$ at right angles. P is kept at rest by the resistance of the curve (R) acting in the direction of the normal at P; the tension of the string (T) acting in the direction PA, and the weight (W) acting in the direction NG.

$$\therefore \frac{W}{T} = \frac{GN}{NP} = \tan GPN = \frac{dy}{dx}$$

$$\therefore T = \frac{W}{\frac{dy}{dx}}.$$

Let $a = \frac{1}{2}$ natural length of the string $= AB$.

$a' = NP = y = \frac{1}{2}$ extended length of the string

then $a' = a\left(1 + \frac{\lambda W}{\frac{dy}{dx}}\right)$

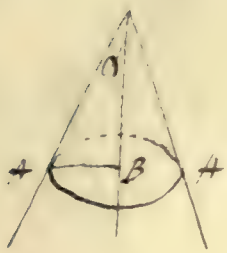
or $(y - a) = \frac{\lambda a W}{\frac{dy}{dx}}$

$$(y-a)\,2\frac{dy}{dx} = 2\lambda a W.$$

$$(y-a)^2 = 2\lambda a W x + C$$

and $y = a$ when $x = 0$ $\therefore C = 0$

wherefore $(y-a)^2 = 2\lambda a W x$ is the eqn $=$?



Let AA' be the position of the string in air

Half of $= m$. w = weight of the string. T its Tension. $2\pi a$ the length of the string before extension. $2\pi a'$ its length after extension, and when in a position of $= m$.

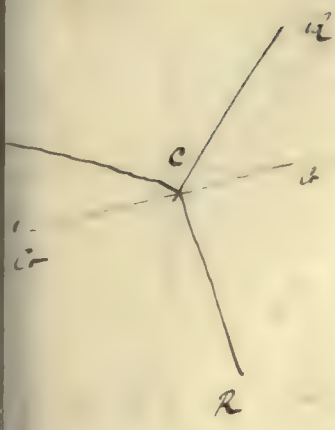
Then $\frac{w}{2\pi a}$ = weight of an unit of length of the string.

$\therefore \frac{w \cos\alpha}{2\pi a}$ = Tension of the string which will be the same throughout.

Therefore $2\pi a' = 2\pi a \left\{ 1 + \frac{1}{2\pi} \frac{w \cos\alpha}{} \right\}$.

or if AB = radius of AA'

$$AB = a\left(1 + \frac{1}{2\pi} w \cos\alpha\right)$$



C the point of application of the forces.

Let C receive a small displacement in direction CQ ⊥ to CR.

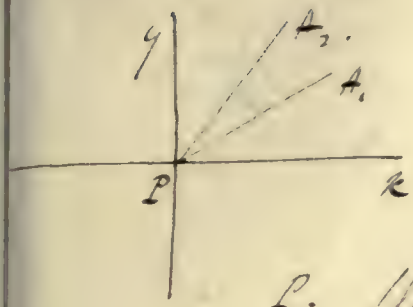
Then by principle of virtual velocities

$Q \cos RCQ = P \cos PCQ$

$R \sin RCR = P \sin PCR$.

similarly $R \sin PCR = R \sin PCR$.

$\therefore P : R :: \sin PCR : \sin PCR : \sin RCR$.



L. be position of the particle, Ly. Lx coordinates, $A_1, A_2 \ldots A_n$ the forces acting upon L, $\alpha_1 \alpha_2 \ldots \alpha_n$ the angles which the direction of these forces make with the axis of x.

Let L receive a small displacement in the direction L x, then by the principle of virtual velocities

$$A_1 \cos\alpha_1 + A_2 \cos\alpha_2 + \ldots - A_n \cos\alpha_n = 0 \quad (1)$$

Let forces each equal to L in magnitude, and acting when placed in the lines LA_1; LA_2 &c - at distances from L equal to $A_1, A_2 \ldots A_n$ respectively

Then for the cent. of g. of this particles supposing it to be placed in the line Ly we should have.

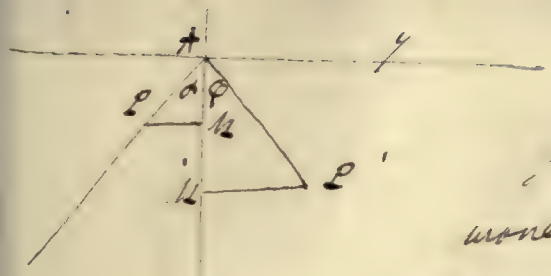
$$A_1 \cos\alpha_1 + A_2 \cos\alpha_2 + \ldots - A_n \cos\alpha_n = 0 \quad (2)$$

But from = n (1) and the = n (2) is true

∴ the c. of g. of this particles is in the line Ly.

Similarly by taking the virtual velocities in the direction of Ly. and estimating the moments of the forces in (2) about the axis of x, we should find their c. of g. in the line Lx and ∴ L is the c. of g.

Therefore L would be at rest if its position were rendered with that of the c. of g. of a number of particles placed in the directions in which the forces act and at distances proportional to the magnitudes of those forces



From the conditions of the question the curve must pass thro' A. Let Ax (vertical) be the axis of x. Ay (horizontal) the axis of y. LL' the position of the weights m, m' at any time t. m, m' their masses. α = inclination of the inclined plane to the vertical $AL = \rho$. $AL' = \rho'$ the distances of L, L' from A. $\angle LAM = \phi$. $AM = x$, $AM' = x'$

Then if L receive a small displacement along LA and L' along the required curve we have by virtual vels.

$$m\,dx + m'\,dx' = 0.$$

Integrate and we have

$$mx + m'x' = C \quad \text{a constant.}$$

$\therefore$ using polar co-ords.

$$m\rho\cos\alpha + m'\rho'\cos\phi = C \qquad (1)$$

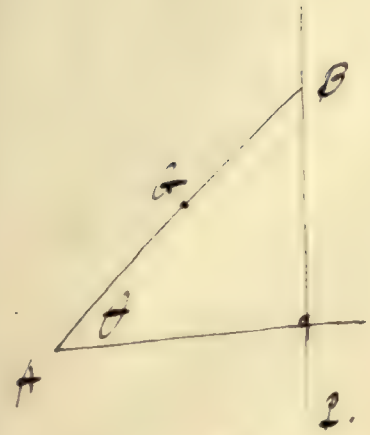
but $\rho + \rho' = l \quad \therefore \rho' = l - \rho$ (2)

substituting in (1) the value of ρ' found from (2)

$$-m(\rho - l)\cos\alpha + m'\rho'\cos\phi = C.$$

$$\rho'(m'\cos\phi - m\cos\alpha) = C - ml\cos\alpha.$$

$$\rho' = \frac{C - ml\cos\alpha}{m'\cos\phi - m\cos\alpha}$$



Let P recede a small displacement. a cos θ

Then Q will recede a disp. $\frac{Qa}{2}\delta\theta$.

$$\therefore P a \cos\theta = \frac{Qa}{2}\delta\theta$$

$$\text{or } \frac{1}{\tan\theta} = \frac{2P}{Q\delta\theta}$$

$$\therefore \frac{m'}{2b}\left\{3b^2+\dot{\rho}^2-4a^2\right)+mx=C$$

$$\frac{m'}{2b}\left\{3b^2+(l-\rho)^2-4a^2\right)+m\rho\cos\theta=C$$

$$m'\left(3b^2+l^2-2l\rho+\rho^2-4a^2\right)+2bm\rho\cos\theta=2bC.$$

$$\rho^2m'+2(2m b\cos\theta-lm')\rho=2bC+(4a^2-3b^2)m'$$

$$\text{or } m'\rho^2+2(2mb\cos\theta-lm')\rho=C$$

where C is a constant.

If $n=1$ the orbit form

$$m\ddot{x}+mx=C$$

whilst here $m'\rho^2+2(2mb\cos\theta-lm')\rho=C$.

Newton generally. $v' - v = e(u - u')$ (1, $m(u - v) = m'(v' - u')$
& a, a' being velocities before impact.

Then $v' - v = a + a'$ for the elasticity is perfect

$$m(a - v) = m'(v' + a').$$

from wh. $v = \frac{(m - m')a}{m + m'} - \frac{2m'a'}{m + m'}$

$$v' = \frac{(m - m')a'}{m + m'} + \frac{2ma}{m + m'}.$$

$\therefore (m+m')^2 mv^2 = m^3a^2 - 2m^2m'a^2 + mm'^2a^2 + 4mm'^2a'a - 4m'm^2a'a + 4m'^2ma'^2$

$(m - m')^2 . m'v'^2 = m'm^2a^2 - 2mm'^2a^2 + m'^3a'^2 + 4m'm^2a'a - 4mm'^2a'a + 4m'^2ma'^2$

$\therefore (m - m')^2 . mv^2 + m'v'^2 = (m - m')^2 . ma^2 + m'a'^2.$

or $mv^2 + m'v'^2 = ma^2 + m'a'^2.$

wherefore sum of the vis vivæ is $ma^2 + m'a'^2$.

même avec le cas exemple.

$$v' - v = e(a - a') \quad ; \quad m(a - v) = m'(v - a')$$

ou $v' - v = e(a \bar{-} a')$; $m(a - v) = m'(v \bar{-} a')$.

$$\therefore v' = v + e(a \bar{-} a') \qquad v = \frac{m(a - v) - m'a'}{m'}$$

$$m'v + m'e(a \bar{-} a') = ma - mv + m'a'$$

$$(m' + m)v = ma + m'a' - m'e.(a \bar{-} a')$$

$$v = \frac{ma + m'a' - m'e(a \bar{-} a')}{m' + m.}$$

$$v' = \frac{ma + m'a' - m'e(a \bar{-} a')}{m' + m} + \frac{(m' + m)(a \bar{-} a')e}{m' + m}$$

$$= \frac{ma + m'a' + me(a \bar{-} a')}{m' + m}$$

$$\therefore (m + m')^2 (mv^2 + m'v'^2) =$$

$$m(m^2a^2 + 2mm'aa' + m'^2a'^2 + m'^2e^2a^2 - 2m'^2e^2aa' + m'^2e^2a'^2$$

$$\mp (ma + m'a')(m'e(a \bar{-} a')$$

$$+ m'(m^2a^2 + 2mm'aa' + m'^2a'^2 + m^2e^2a^2 - 2m^2e^2aa' + m^2e^2a'^2$$

$$\mp (ma + m'a')me(a \bar{-} a').$$

$$= m^3a^2 + 2m^2m'aa' + mm'^2a'^2 + m'm^2a^2 + 2mm'^2aa' + m'^3a'^2$$

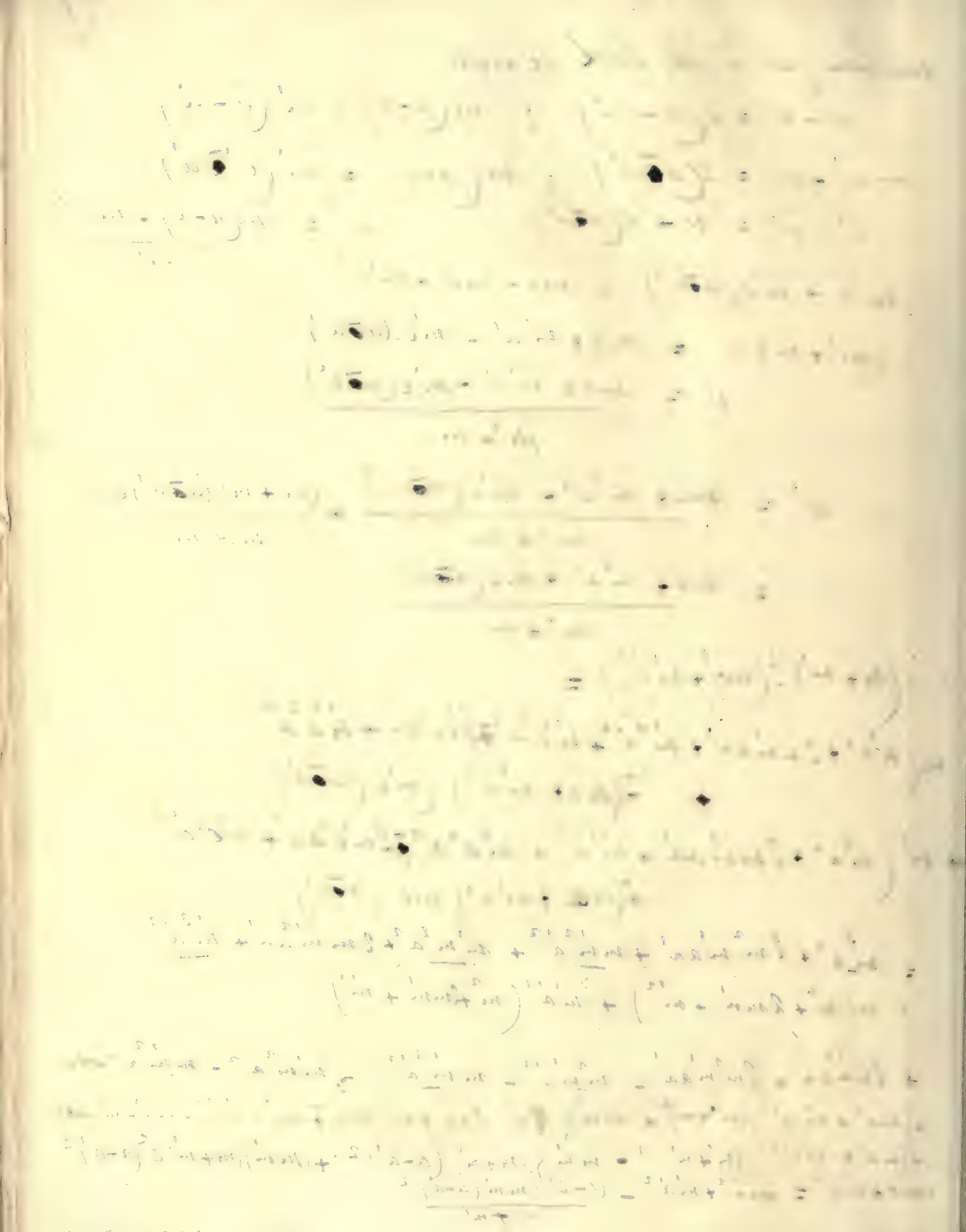
$$= ma^2(m^2 + 2mm' + m'^2) + m'a'^2(m^2 + 2mm' + m'^2)$$

$$+ 2mm'^2aa' + 2m^2m'aa' - m^2m'a'^2 - mm'^2a'^2 - m'm^2a^2 - mm'^2a^2 + 4e$$

$$= (ma^2 + m'a'^2)(m^2 + m'^2) + mm'\{2m.2aa' + m.2aa' - ma'^2 - m'a'^2 - ma^2 - m'a^2\} + 4e$$

$$+ (ma^2 + m'a'^2)(m'^2 + m^2) - mm'\{(m + m')(a - a')^2 + (mm')(m + m')e^2(a - a')^2$$

$$mv + m'v' = ma^2 + m'a'^2 - \frac{(1 - e^2)mm'(a - a')^2}{m + m'}$$



we have $(v'-v) = e(a-a')$.

$m(a-v) = m'(v'-a')$. { supposing the balls to move in the same direction

$\therefore v' = \dfrac{e(a-a') + v}{m'}$

$= \dfrac{m(a-v) + m'a'}{m'}$

$m'e(a-a') + m'v = ma - mv + m'a'$

$\therefore v = \dfrac{ma + m'a' - m'e(a-a')}{m+m'}$

$v' = \dfrac{ma + m'a' + m'e(a-a')}{m+m'}$ (1)

But by the question $v' = 0$

$\therefore ma + m'a' + m'ea = m'a'e$.

or $(m' - m'e)\,a' = -(m'e + m)\,a$.

or $m'a'(e-1) = (m'e + m)\,a$.

If $m' = m$ we have from (1)

$a + a' + ae - a'e = 0$

$\therefore a(1-e) = -a(1+e)$

or a' must be a velocity in the opposite direction $= \dfrac{1+e}{1-e}$

$a' = -\left(\dfrac{1+e}{1-e}\right)a$

If v. v' be the velocities after collision we have

$$v = \theta = \frac{ma + m'a' - m'e(a-a')}{m+m'} \quad (1)$$

$$v' = \phi = \frac{ma + m'a' + me(a-a')}{m+m'} \quad (2)$$

from (2) we have $ma + m'a' = -me(a-a')$.

$\therefore$ substituting in (1) we have $\frac{(m+m')e(a-a')}{m+m'} = -\beta$.

$$ea - ea' = -\beta.$$

$$a = \frac{ea' - \beta}{e} \quad (1).$$

Also from (2). $ma + m'a' + mea - mea' = 0$.

$$\therefore m(1+e)a = (me - m')a'$$

$$\therefore a = \frac{(me - m')a'}{m(1+e)} \quad (2).$$

Equating (1) and (2) we have.

$$\frac{(me - m')a'}{m+me} = \frac{a'e - \beta}{e}$$

$$(me - m')a'e = (me + me^2)a' - m(1+e)\beta$$

$$m(1+e)\beta = (me^2 + me)a'$$

$$a' = \frac{m\beta(1+e)}{me^2 + m'e}$$

$$a = \frac{m\beta(1+e)}{me + m'e} - \frac{\beta(m+m')}{(m+m')e}$$

$$= + \frac{me\beta - m'\beta}{(m+m')e} = \frac{(me - m')\beta}{(m+m')e}$$

$=$

In this case if v' = velocity of B after impact.

v = ——— A ——— before imp.

a = ——— — ———

$m.m'$ the masses of A & B respectively

$v' - v = e(a)$ $\therefore v' = ea + v$.

$m.(a-v) = m'(v')$ $\therefore v' = \frac{m(a-v)}{m'}$

$\therefore m'ea + m'v = ma - mv$.

$(m' + m)v = ma - m'ea$

$\therefore v = \frac{ma - m'ea}{m+m'}$

Also $v' = \frac{ma - m'ea}{m+m'} + \frac{(m+m')ea}{m+m'}$

$= \frac{ma + mea}{m+m'}$

$\therefore \frac{m'ma}{m+m'}(1+e)$ = moment of B after impact.

Let m'' be the required mass of the 3rd body.

Then. $m''a = \frac{mm'a(1+e)}{m+m'}$

$\therefore m'' = \frac{mm'}{m+m'}(1+e)$.

a = velocity of A before impact. v' = velocity of B after impact.
v = " " A after " m, m' the masses of A & B.

Then $v' - v = a$ $\therefore v' = a + v$.

$m(a - v) = m'v'$ $\therefore v' = \frac{m(a-v)}{m'}$.

$\therefore m'a + m'v = ma - mv$

$\therefore (m' + m)v = ma - m'a$

$$v = \frac{ma - m'a}{m + m'}.$$

$$v' = \frac{ma + ma}{m + m'}$$

Also $\frac{ma - m'a}{m + m'} = 0$ $\therefore m = m'$ (1)

Also $\frac{ma + ma}{m + m'} = \frac{a}{n}$.

$\therefore \frac{m}{m + m'} \cdot (1 + e) = \frac{1}{n}$.

or $\frac{m'e(1+e)}{m'(1+e)} = \frac{1}{n}$.

$\therefore ne = 1$ or $e = \frac{1}{n}$.

$\frac{m}{m'} = e = \frac{1}{n}$

or $\frac{m'}{m} = n$.

Retaining the notation as above.

$$v' - v = (a + a') \quad \therefore v' = a + a' + v.$$

$$\text{also } m(a - v) = m'(v' + a') \quad \therefore v' = \frac{m(a-v) - m'a'}{m'}$$

$$\therefore ma + ma' + m'v = ma - mv - m'a'$$

$$(m + m')v = ma - ma - 2m'a'$$

$$v = \frac{ma - m'a - 2m'a'}{m + m'}$$

$$\text{also } v' = \frac{(m - m')a' + 2ma}{m + m'}$$

And by the question $\frac{(m - m')a' + 2ma}{m + m'} = 0$

$$\therefore m(a' + 2a) = m'a'$$

$$\therefore \frac{m}{m'} = \frac{a'}{a' + 2a}$$

or since the velocities are equal

$$\frac{m}{m'} = \frac{1}{3}.$$

$$\text{or } m' : m :: 3 : 1$$

To find the velocity of the second sphere.

$$v' - v = ea \quad \therefore v' = ea + v$$

$$m(a - v) = m'v' \quad \therefore v' = \frac{m(a-v)}{m'}$$

$$\therefore m'ea + m'v = ma - mv$$

$$v = \frac{(m - m'e)a}{m + m'} \qquad v' = \frac{(m - m'e)a + (m + m')ea}{m + m'}$$

$$= \frac{(m + me)a}{m + m'} = \frac{ma(1+e)}{m'+m}$$

To find velocity of 3rd sphere.

$$v'' - v' = \left(\frac{(1+e)ma}{m+m'}\right)e \quad \therefore v'' = \left(\frac{(1+e)mae}{m+m'}\right) + v'$$

$$m''\left(\frac{(1+e)ma}{m+m'} - v'\right) = m''v'' \quad \therefore v'' = \frac{m'}{m''}\left(\frac{(1+e)ma}{m+m'} - v'\right)$$

$$\therefore \frac{(1+e)mae + (m+m')v'}{m+m'} = \frac{m'}{m''}\left(\frac{(1+e)ma - (m+m')v'}{m+m'}\right)$$

$$(1+e)m''mae + m''(m+m')v' = (1+e)m'ma - m'(m+m')v'$$

$$\therefore (m'' + m')(m+m')v' = ma(1+e)(m' - m''e)$$

$$v' = \frac{ma(1+e)(m' - m''e)}{(m'' + m')(m+m')}$$

$$v'' = \frac{ma(1+e)(m' - m''e) + (m'' + m')e \cdot ma(1+e)}{(m'' + m')(m' + m)} = \frac{m'm(1+e)^2 a}{(m'' + m')(m' + m)}$$

Similarly $v''' = \dfrac{m''m'm(1+e)^3 a}{(m''' + m'')(m'' + m')(m' + m)}$

$$v^n = \frac{m^n m^{n-1} \cdots m''m'm(1+e)^n a}{(m^n + m^{n-1})(m^{n-1} + m^{n-2}) \cdots (m' + m)}.$$

Let t_2. t_3 & t_n be the time that elapses before the 2nd. 3rd &c. spheres are put in motion.

Then $t_2 = \frac{a}{\beta}$.

also $v_2 = \frac{\beta}{2}$

$\therefore t_3 - t_2 = \frac{2a}{\beta}$

again $v_3 = \frac{2\beta}{3\,2} = \frac{\beta}{3}$ $\therefore t_4 - t_3 = \frac{3a}{\beta}$.

$v_{n-1} = \frac{\beta}{n-1}$ $\therefore t_n - t_{n-1} = \frac{(n-1)a}{\beta}$.

By addition $t_n = \left(1.2.3. \text{ &c } (n-1)\right) \frac{a}{\beta}$.

$= \frac{n(n-1)}{2} \cdot \frac{a}{\beta}$.

Let m denote the mass, and a the velocity of the impinging sphere.
m_1, m_2, v_1, v_2 &c. the mass and velocities of the 1st. 2nd. intermediate spheres.

Then $v_2 = \dfrac{4m_2 m_1}{(m+m_1)(m_1+m_2)} = \dfrac{4m_2 a}{\left(\frac{m}{m_1}+1\right)(m_1+m_2)}$.

Now v_2 must be a maxim. $\therefore \left(\frac{m}{m_1}+1\right)(m_1+m_2)$ a minim

Different. and reduce.

$$\frac{m}{m_1}+1 - \frac{m}{m_1^2}(m_1+m_2) = 0$$

$$mm_1 + m_1^2 - mm_1 - mm_2 = 0 \quad \therefore m_1^2 = mm_2$$

$$m_1 = \sqrt{mm_2}.$$

Similarly it might be shewn that $m_2 = \sqrt{m_1 m_3}$

$$m_3 = \sqrt{m_2 m_4}$$

$$\&c. \quad \&c.$$

$$m_{n-1} = \sqrt{m_{n-2} m'}$$

$\therefore$ the intermediate spheres must be geometrical means betw the original spheres.

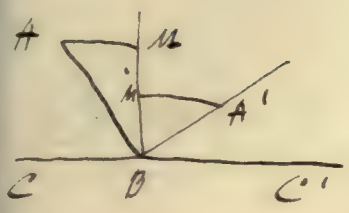
also $v_2 = \dfrac{4mm_1 a}{(m+m_1)(m_1+m_2)} = \dfrac{4m\sqrt{mm_2}\cdot a}{(m+\sqrt{mm_2})(\sqrt{mm_2}+m_2)}$ or if $r = \left(\frac{m}{m'}\right)^{\frac{1}{n+2}}$.

$$v' = \frac{4am}{(\sqrt{m}+\sqrt{m_2})(\sqrt{m}+\sqrt{m_2})} = \frac{4am}{m+2\sqrt{mm_2}+m_2} = \frac{4a\left(\frac{m}{m_1}\right)^{\frac{2}{n+2}}}{\left(\frac{m}{m_1}\right)^{\frac{2}{n+2}} + 2\left(\frac{m}{m_2}\right)^{\frac{1}{n+2}} + 1} = \frac{4a\left(\frac{m}{m_1}\right)^{r^2}}{\&c.}$$

$$= \frac{4ar^2}{r^2+2r+1} = \frac{4r^2}{(r+1)^2}a \qquad \text{also } v^3 = \frac{8r^3 a}{(r+1)^3}.$$

$$v^{n+2} = \frac{2^{n+2}\cdot\left(\frac{m}{m_1}\right)}{(r+1)^{n+2}} = \frac{2^{n+2}\left(\frac{m}{m_1}\right)}{\left(\frac{m}{m_1}\right) + (n+2)\left(\frac{m}{m_1}\right)^{\frac{n+1}{n+2}} + \&c. + 1}$$

(now the sum of all the coefficients in the latter expansion $= 2^{n+2}$ and as $\left(\frac{m}{m_1}\right)$ is nearly unity all its powers are nearly unity $\therefore v^{n+2} = \left(\frac{m}{m_1}\right)^{\frac{1}{2}} a$. Taking $\left(\frac{m}{m_1}\right)^{\frac{1}{2}}$ as the mean betwixt $\frac{m}{m_1}$ and unity.



A B the direction of incidence.

$\angle$ A B C = θ. resolve velocity // and $\perp$ to plane. the former will not be altered. the latter will be diminished in the prop. of 1 : $\frac{1}{\sqrt{3}}$

$\therefore$ if B M' = $\frac{BM}{\sqrt{3}}$ and A' M' = A M. B A' is the direction after incidence. = θ'

Now $\frac{\tan\theta}{\tan\theta'} = \frac{BM}{BM'} = \sqrt{3}$ $\therefore \tan\theta = \sqrt{3}\tan\theta'$

Let v = velocity before ~~ref~~ incidence v' = velocity after &c.

Then $v' = \sqrt{v^2\sin^2\theta + \frac{v^2\cos^2\theta}{3}} = \frac{v}{\sqrt{3}}\sqrt{3\sin^2\theta + \cos^2\theta} = \frac{v}{\sqrt{2}}$

$\therefore \frac{2\sin^2\theta + 1}{3} = \frac{1}{2}$ or $4\sin^2\theta + 2 = 3$

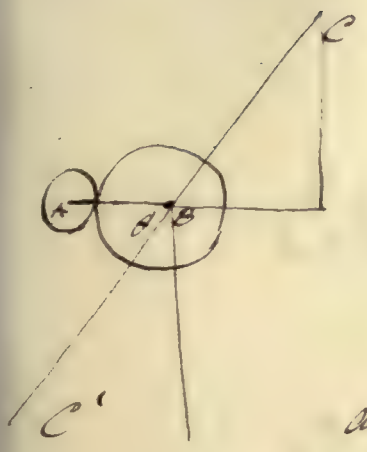
$\therefore \sin^2\theta = \frac{1}{4}$ $\sin\theta = \frac{1}{2}$

or $\theta = \frac{\pi}{6}$

$\therefore \tan\theta = \frac{\sqrt{3}}{\sqrt{3}} = \sqrt{3} = \sqrt{3}\tan\theta'$

wherefore $\tan\theta' = 1$

or $\theta' = \frac{\pi}{4}$.



Let AB be the line joining the centres of the spheres. m, m' the masses of the moving and of the stationary sphere. v = velocity of B (the former) Resolve velocities ∥ and ⊥ to AB. the latter will not be affected by the collision

Now velocity of B in direction BA = $v \cos\theta$ - before collision

$$= \frac{m \cdot v \cos\theta}{m+m'} \text{ after collision}$$

for the spheres are inelastic

∴ if v' represent the velocity of B after collision

$$v'^2 = v^2 \sin^2\theta + \frac{m^2 v^2}{(m+m')^2} \cos^2\theta$$

$$= v^2 \left(1 - \left(\frac{2mm' + m'^2}{(m+m')^2}\right) \cos^2\theta\right).$$

The part of A's velocity effected by impact is $v \cos\alpha$. Let v' v'' be the velocities of the spheres A, B after impact. If $\perp$ BB'.

Then $v' - v'' = e(v + v\cos\alpha)$

$\therefore v'' = v'' + ve(1+\cos\alpha)$

Also since the masses are equal.

$v - v'' = v' + v\cos\alpha$. $\therefore v' = v(1-\cos\alpha) - v''$

$\therefore v'' + ve(1+\cos\alpha) = v(1-\cos\alpha) - v''$

$2v'' = v(1-e) - (v\cos\alpha)(1+e)$

$\therefore v'' = \frac{1}{2}\{v(1-e) - (v\cos\alpha)(1+e)$

$v' = \frac{1}{2}\{v(1-e) + (v\cos\alpha)(1+e) + ve(1+\cos\alpha)$

$v'' = \frac{1}{2}\{v(1+e) - (v\cos\alpha)(1-e)\}$.

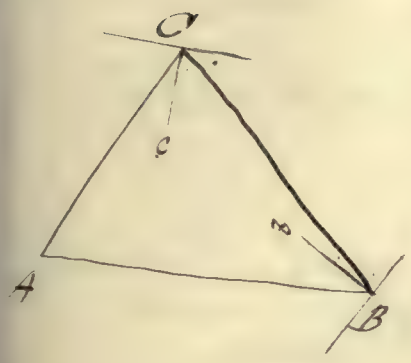
$\therefore v''^2 = \frac{1}{4}v^2\{1+e - \cos\alpha(1-e)\}^2$.

Also if v''' = the whole of A's velocity after impact.

$v'''^2 = \frac{1}{4}v^2\{1-e+\cos\alpha(1+e)\}^2 + v^2\sin^2\alpha$.

Eliff. the Cambridge Exam. mechanics $(1-e^2) = (\frac{e^2}{2} + 2e - 3)\cos\alpha$ formerly

$\therefore \cos\alpha = \frac{1+e}{3+e}$



Let $\alpha.\beta.\gamma$ be the angles at the points A.B.C of the Δ.. Bb, Cc $\perp$ to AC. AB. Then $\angle ABb = \left(\frac{\pi}{2} - \alpha\right)$
$\angle bBC = \left(\beta + \alpha - \frac{\pi}{2}\right)$ $\angle CCB = \frac{\pi}{2} - \beta$, $\angle cCA = \gamma + \beta - \frac{\pi}{2}$

let a = A's velocity at first. Then the stack effected by impact is $a \sin\alpha$.
Let a' = A's velocity in direction Bb after impact.. a'' = B's velocity
Then. $a'' - a' = a \sin\alpha$ $\therefore$ $a'' = a' + a\sin\alpha$.

also $m(a\sin\alpha - a') = m'a''$ $\therefore$ $a'' = \frac{m}{m'}a\sin\alpha - a'$

$\therefore$ $m'a' + m'a\sin\alpha = ma\sin\alpha - ma'$

$\therefore$ $(m' + m)a' = (m - m')a\sin\alpha$ for $\beta = 1$

$$a' = \frac{(m - m')a\sin\alpha}{m + m'}$$

$\therefore$ The whole (velocity)^2 of A after reflection is

$$\frac{\left(m^2 - 2mm' + m'^2\right)a^2\sin^2\alpha + \left(m^2 + 2mm' + m'^2\right)a^2\cos^2\alpha}{(m+m')^2}$$

$$v = \sqrt{\frac{\left(m^2 + m'^2\right)a^2 + 2mm'\cos 2\alpha\, a^2}{(m+m')^2}} = \frac{a}{m+m'}\cdot\sqrt{m^2 + m'^2 + 2mm'\cos 2\alpha}.$$

also $\tan\gamma = \frac{a\cos\alpha\,(m+m')}{a\sin\alpha\,(m - m')}$

$$= \cot\alpha\cdot\frac{m+m'}{m-m'} \qquad (1)$$

again. the velocity of A after impact upon C in the direction AB is $\sqrt{(m^2+m'^2) + 2mm'\cos 2\alpha}$. $\frac{a\cos\beta}{m+m'}$.
and that in the direct. $\perp$ to AB is $\frac{(m-m')a\sin\beta}{(m+m')(m+m')}\cdot\sqrt{m^2+m'^2+2mm'\cos 2\alpha}$

$\therefore$ the whole (velocity)^2 of A after collision with C is

$$= \left(\frac{(m^2 + m'^2) + 2mm' \cos 2\alpha}{(m+m')^2} \right) \left\{ \cos^2\beta + \frac{(m-m'')^2 \sin^2\beta}{(m+m'')^2} \right\} .$$

$$= x^2 \left\{ \left(\frac{m^2 + m'^2 + 2mm' \cos 2\alpha}{(m+m')^2} \right) \left\{ \frac{m^2 + m''^2 + 2mm'' \cos 2\beta}{(m+m'')^2} \right) \right.$$

$$\therefore v = a \sqrt{\left\{ \frac{(m^2 + m'^2) + 2mm' \cos 2\alpha}{(m+m')} \right\} \left\{ \frac{m^2 + m''^2 + 2mm'' \cos 2\beta}{(m+m'')} \right\}}$$

Also we must have $\cot\alpha = \frac{\cos\beta}{\sin\beta} \cdot \left(\frac{m+m''}{m-m''} \right)$

$$\therefore (m - m'') \cot\alpha = (m + m'') \cos\beta .$$

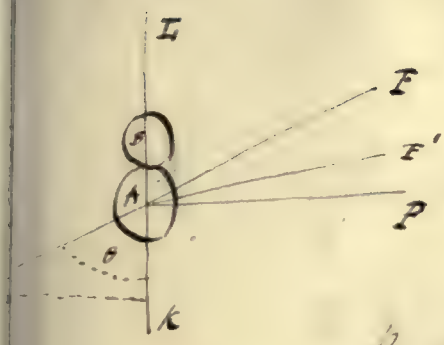
$$m(\cot\alpha - \cos\beta) = m''(\cos\beta + \cot\alpha) .$$

$$\therefore m \sin(\beta - \alpha) = m'' \sin(\beta + \alpha)$$

$$= m'' \sin\gamma .$$

$$\therefore \frac{m}{m''} = \frac{\sin\gamma}{\sin(\beta - \alpha)} .$$

$$\text{Also } \frac{m}{m'} = \frac{\sin\beta}{\sin(\alpha - \gamma)} \text{ from (1).}$$



By Law (9). the velocity of A // to kL after impact will be. $\frac{1}{2}\cdot\left\{v(1+e)+v\cos\theta(1+e)\right\}$ or $+\frac{v}{2}\cos\theta(1+e)$ since the velocity of B is nothing.

also the velocity $\perp$ to kL is

$v\sin\theta$

$\therefore \tan F'AP = \dfrac{\frac{v}{2}\cos\theta(1+e)}{v\sin\theta} = \dfrac{1+e}{2}\cdot\cot\theta.$

$\therefore \tan FAF' = \dfrac{\cot\theta - \frac{1+e}{2}\cot\theta}{1+\frac{1+e}{2}\cot^2\theta} = \dfrac{\frac{1-e}{2}\cot\theta}{1+\frac{1+e}{2}\cot^2\theta}$

if FAF' must be a max. $\tan FAF'$ must be a max.

$\therefore \dfrac{2}{3+e}\left\{\sec^2\theta + \dfrac{1+e}{2}\operatorname{cosec}^2\theta\right\} = 0 \;\div$

$\dfrac{1}{\cos^2\theta} + \dfrac{1+e}{2}\cdot\dfrac{1}{\sin^2\theta} = 0.$

$\therefore \left(1+\dfrac{1+e}{2}\cot^2\theta\right)\left(\dfrac{1-e}{2}\right)\operatorname{cosec}^2\theta + \left(\dfrac{1-e}{2}\right)\cot^2\theta\cdot(1+e)\operatorname{cosec}^2\theta = 0.$

or $(1+e)\cot^2\theta = 1+\left(\dfrac{1+e}{2}\right)\cot^2\theta.$

$\left(\dfrac{1+e}{2}\right)\cot^2\theta = 1 \quad \therefore \cot^2\theta = \dfrac{2}{1+e}$

$\therefore \tan FAF' = \dfrac{\frac{2\sqrt{2}}{\sqrt{1+e}}\cdot\left(\frac{1-e}{2}\right)}{2+(1+e)\frac{2}{1+e}} = \dfrac{\sqrt{2}\cdot\frac{1-e}{\sqrt{1+e}}}{4} = \dfrac{\frac{1}{2}\cdot(1-e)}{(2\cdot(1+e))^{\frac{1}{2}}}$

smalles having been taken equal $\therefore n=1$

or $\tan\theta = \left(\dfrac{n+e}{n+1}\right)^{\frac{1}{2}}$. $\tan\varphi = \dfrac{\frac{1}{2}(1-e)}{\{(n+1)(n+e)\}^{\frac{1}{2}}}$.

The nth a function will be

$$\frac{d^2x}{t} = -\mu x^n.$$

Multiply by $2\frac{dx}{t}$

$$\therefore 2\frac{dx}{t}\frac{d^2x}{t} = -2\mu . x^n \frac{dx}{t}.$$

$$\left(\frac{dx}{t}\right)^2 = -\frac{2\mu}{n+1} x^{n+1} + C.$$

at the commencement of the motion $x = \dot{x}$.

$$\therefore 0 = -\frac{2\mu}{n+1}\dot{x}^{n+1} + C$$

$$\text{or } C = \frac{2\mu}{n+1}\dot{x}^{n+1}$$

at a distance a from the centre of force where

$$\left(\frac{dx}{t}\right)^2 = -\frac{2\mu}{n+1} a^{n+1} + C$$

$$= \frac{2\mu}{n+1}\left(\dot{x}^{n+1} - a^{n+1}\right.$$

$$\therefore \text{from the graph } \beta^2 = \frac{2\mu}{n+1}\left(\dot{x}^{n+1} - a^{n+1}\right)$$

$\therefore$ $\mu = n \times$ proton ii

$$\frac{d^2x}{t} = -\frac{\mu}{x^3}$$

$$\therefore \frac{2dx}{t}\frac{d^2x}{t} = -\frac{2\mu}{x^3}\frac{dx}{t}$$

$$\left(\frac{dx}{t}\right)^2 = +\frac{2\mu}{2x^2} = \frac{\mu}{x^2} + C.$$

$$\therefore C = -\frac{\mu}{x^2}.$$

let a = initial distance

$$\therefore 0 = \frac{2\mu}{2a} + C = \frac{\mu}{a^2} + C$$

$$\therefore \left(\frac{dx}{t}\right)^2 = \mu\left\{\frac{1}{x^2} - \frac{1}{a^2}\right\} = \mu\frac{(a^2 - x^2)}{a^2x^2}.$$

$$\text{or } -\frac{dt}{x} = \frac{a}{\sqrt{\mu}} \cdot \frac{x}{\sqrt{a^2 - x^2}}$$

$$\therefore t = +\frac{a}{\sqrt{\mu}}\left(a^2 - x^2\right)^{\frac{1}{2}} + C$$

when $t = 0$ $x = a$ $\therefore$ $C = 0$.

$$\text{or } t = \frac{a}{\sqrt{\mu}}\left(a^2 - x^2\right)^{\frac{1}{2}}$$

when the particle arrives at the cent. $x = 0$.

$$\therefore t = \frac{a^2}{\sqrt{\mu}}$$

The eqn of motion will be.

$$\frac{d^2x}{t} = -\frac{\mu}{x^2}.$$

$$2\left(\frac{dx}{t}\right)\frac{d^2x}{t} = -\frac{2\mu}{x^2}\frac{dx}{t}.$$

$$\left(\frac{dx}{t}\right)^2 = \frac{2\mu}{x} + C.$$

when $\left(\frac{dx}{t}\right)^2 = 0 \quad x = \infty$

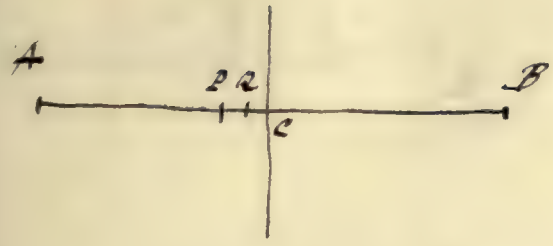
$$\therefore \quad 0 = 0 + C \text{ or } C = 0.$$

If $x = a$ mean

$$\left(\frac{dx}{t}\right)^2 = \frac{2\mu}{a}$$

$$\text{or } v^2 = \frac{2\mu}{a}.$$

$$\therefore v = \sqrt{\frac{2\mu}{a}}.$$



A. B. the positions of the centres of force. C the middle pt. of the line joining their centres.

$AC = CB = b$. ; Q the initial posit^n of the body. $QC = a$. R the posit^n of the particle at the expiration of any time t.

Then the eq^n of motion is

$$\frac{d^2x}{t} = \mu\{AR) - 2\mu RC.$$

$$= -2\mu x.$$

$$\left(\frac{dx}{t}\right)^2 = -2\mu x^2 + C.$$

$$0 = -2\mu a^2 + C \quad \therefore C = 2\mu a^2$$

$$\therefore \left(\frac{dx}{t}\right)^2 = 2\mu.(a^2 - x^2).$$

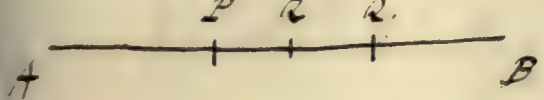
$$\therefore \frac{dt}{a} = \frac{1}{\sqrt{2\mu}} \frac{1}{\sqrt{a^2 - x^2}}$$

$$t = \frac{1}{\sqrt{2\mu}} \cdot \cos^{-1}\frac{x}{a} + C \quad (1)$$

when $t = 0$ $x = a$. and the corresp^g angle $= 0.\pi.2\pi$.

$\therefore$ the time of oscillat$^n = \dfrac{\pi}{\sqrt{2\mu}}$. (2)

Also from (1). $t\sqrt{2\mu} = \cos^{-1}\frac{x}{a} \quad \therefore x = a\cos(t\sqrt{2\mu})$



A. B. the centres of force. P the point of proj.ⁿ AP = 2a. BP = 2a', R the neutral point of attⁿ. R' the pointn. of the body at any time t. μ, μ' the absolute force of attⁿ. at A, B. V = velocity of proj.ⁿ AR' = x.

$$\frac{d^2x}{dt^2} = -\left\{\frac{\mu}{x^2} + \frac{\mu'}{(2a+2a'-x)^2}\right\}$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = +\frac{2\mu}{x} + \frac{2\mu'}{(2a+2a'-x)} + C$$

initially when

$$V^2 = \frac{2\mu}{2a} + \frac{2\mu'}{2a} + C = \frac{\mu+\mu'}{a} + C.$$

$$\therefore C = V^2 - \frac{\mu+\mu'}{a}.$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} + \frac{2\mu'}{(2a+2a'-x)} + V^2 - \frac{\mu+\mu'}{a}$$

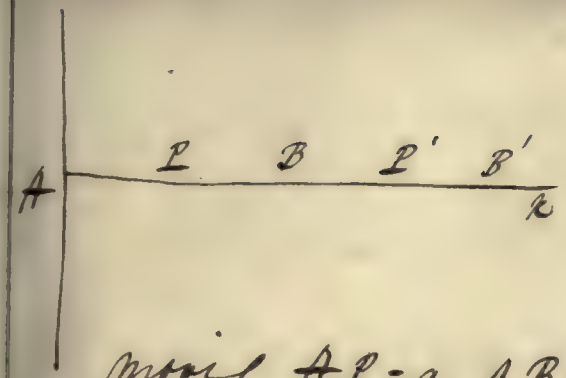
at the neutral point of att.ⁿ we have. $\frac{\mu}{x^2} = \frac{\mu'}{(2a+2a'-x)^2}$

$$\therefore \frac{\sqrt{\mu}}{x} = \frac{\sqrt{\mu'}}{2a+2a'-x}$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = 0 = V^2 - \frac{\mu+\mu'}{a} + \frac{\mu+\sqrt{\mu\mu'}}{(a+a')} + \frac{\mu'+\sqrt{\mu\mu'}}{a+a'} \qquad \text{or } x = \frac{2(a+a')\sqrt{\mu}}{\sqrt{\mu}+\sqrt{\mu'}}$$

$$= V^2 - \frac{\mu}{a} - \frac{\mu'}{a'} + \frac{(\sqrt{\mu}+\sqrt{\mu'})^2}{a+a'}$$

$$\therefore V^2 = \frac{\mu}{a} + \frac{\mu'}{a'} - \frac{(\sqrt{\mu}+\sqrt{\mu'})^2}{a+a'}.$$



A the origin of co ord^{s}. Ax the axis of x being the line along wh. the card of ce. and the particle are moving. AL = a. AB = b the mutual positions of the body, and card. off ce. AL' = x the posn. of the part. at the expiration of the time t. AB' = c' the posn. of the card. off ce. at expiration of time t. u = mutual velocity of L. v = comm. vel. of C.

Then $-\frac{d^2x}{dt^2} = \mu x'$ where $x' = L'B' = AB' - AL'$

$= b + vt - x$

$\therefore \frac{d^2x}{dt^2} = \mu(b + vt - x)$

$\therefore \frac{d^2}{dt^2}(b + vt - x) + \mu(b + vt - x) = 0$

By integ. we have

$x = A \cos \sqrt{\mu}\, t + B \sin \sqrt{\mu}\, t + vt + b.$

in wh. A and B are arbitrary Constts.

$\frac{dx}{dt} = -\sqrt{\mu} A \sin \sqrt{\mu}\, t + \sqrt{\mu} B \cos \sqrt{\mu}\, t + v.$

when $t = 0$ $x = a$, $\frac{dx}{dt} = u$

$\therefore a = A + b \therefore A = a - b.$

$\frac{dx}{dt} u = \sqrt{\mu} B + v \therefore B = \frac{u - v}{\sqrt{\mu}}.$

$\therefore x = b + vt + \frac{u - v}{\sqrt{\mu}} \sin \sqrt{\mu}\, t + (a - b) \cos \sqrt{\mu}\, t.$

The notation remaining as in the last example we have

$$\frac{d^2x}{dt^2} = -\mu\{b+vt-x\}.$$

$$\therefore \frac{d^2x}{dt^2} + \mu(b+vt-x) = 0$$

$$\text{or } \frac{d^2}{dt^2}(b+vt-x) - \mu(b+vt-x) = 0.$$

$$\therefore x = b+vt + Ae^{t\sqrt{\mu}} + Be^{-t.\sqrt{\mu}}.$$

when $t=0$ $x = a$.

$$\therefore a = b + A + B \quad \text{or} \quad A+B = a-b.$$

$$\frac{dx}{dt} = v + \sqrt{\mu}.Ae^{t\sqrt{\mu}} - B\sqrt{\mu}.e^{-t\sqrt{\mu}}.$$

when $t=0$ $\frac{dx}{dt} = u$

$$\therefore \frac{u-v}{\sqrt{\mu}} = (A-B) \quad \therefore A = \frac{1}{2}\left\{a-b+\frac{u-v}{\sqrt{\mu}}\right\}$$

$$B = \frac{1}{2}\left\{a-b-\frac{u-v}{\sqrt{\mu}}\right\}.$$

$$\therefore x = b+vt + \left\{\sqrt{\mu}(a-b)+u-v\right\}\frac{e^{t\sqrt{\mu}}}{2\sqrt{\mu}} + \left\{\sqrt{\mu}.(a-b)-(u-v)\right\}\frac{e^{-t\sqrt{\mu}}}{2\sqrt{\mu}}$$

Answer.

+ the origin + the direction of motion $AL' = a'$ $AL = a$ the
similar position of the corresponding particle. AM' $AM = x$ position
of the corresponding particle at the expiration of t seconds.

The eqn of motion is $\frac{d^2x}{dt^2} = -\mu MM'$.

$$= \mu\left(a' + \beta t + \tfrac{1}{2} f t^2 - x\right)$$

Now $\frac{d}{dt}\left(a' + \beta t + \tfrac{1}{2} f t^2 - x\right) = \beta + ft - \frac{dx}{dt}$

$$\frac{d^2}{dt^2}\left(a' + \beta t + \tfrac{1}{2} f t^2 - x\right) = f - \frac{d^2x}{dt^2} \quad \therefore \frac{d^2x}{dt^2} = \frac{d^2}{dt^2}\left(a' + \beta t + \tfrac{1}{2} f t^2 - x\right) + f$$

$$\therefore \frac{d^2}{dt^2}\left(x' + a' - \beta t - \tfrac{1}{2} f t^2\right) + \mu\left(x - a' - \beta t - \tfrac{1}{2} f t^2\right) + f = 0$$

$$x = \sqrt{\mu}\, A \cos\sqrt{\mu}\, t + B \sin\sqrt{\mu}\, t + a' + \beta t + \tfrac{1}{2} f t^2 - \frac{f}{\mu} = 0$$

$$\frac{dx}{dt} = -\sqrt{\mu}\, A \sin\sqrt{\mu}\, t + \sqrt{\mu}\, B \cos\sqrt{\mu}\, t + \beta + ft = 0.$$

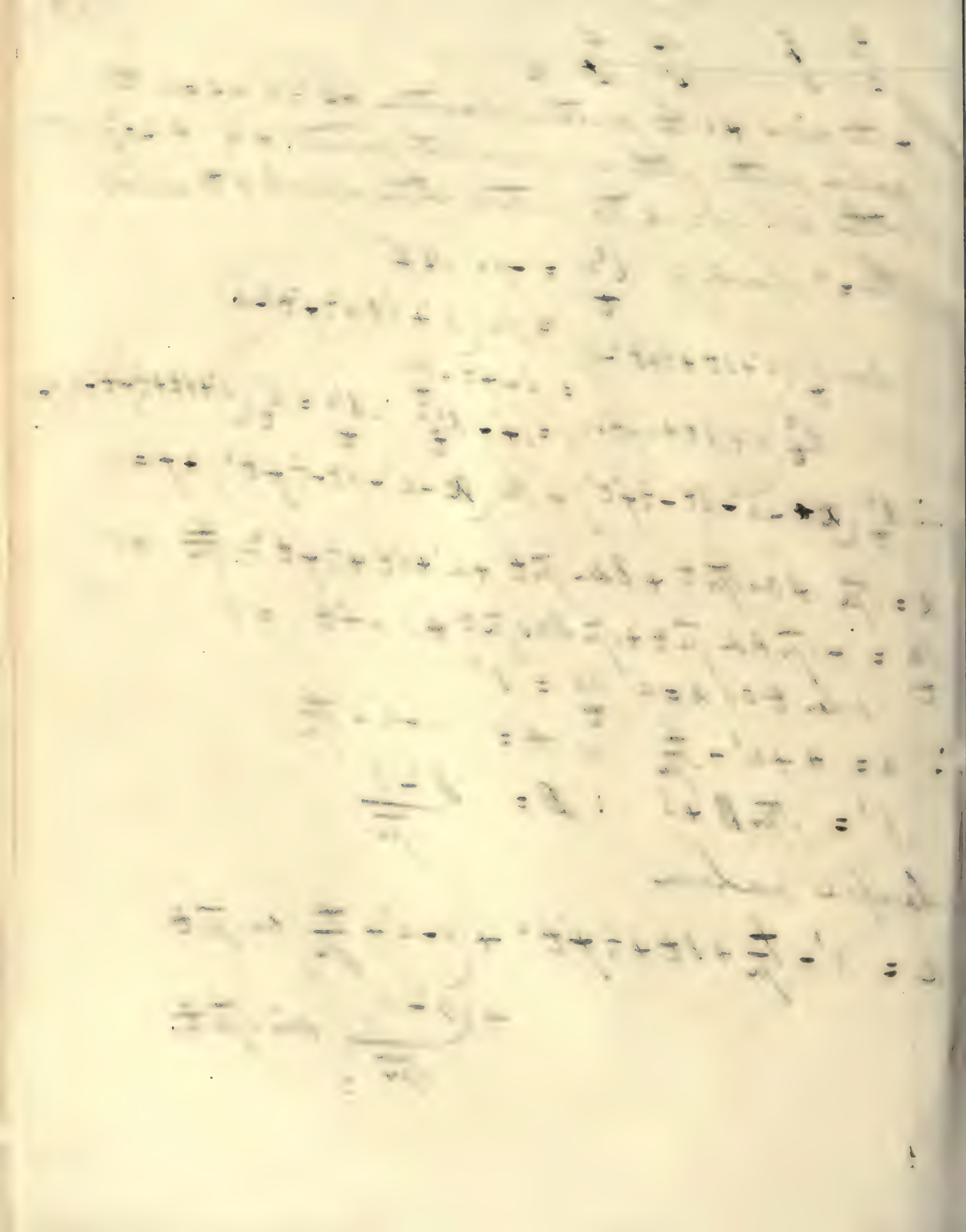
when $t = 0$ $x = a$ $\frac{dx}{dt} = \beta'$

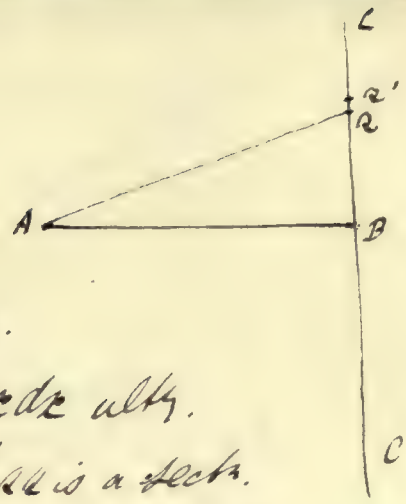
$$\therefore a = A + a' - \frac{f}{\mu} \quad \text{or} \quad A = a - a' + \frac{f}{\mu}.$$

$$\beta' = \sqrt{\mu}\, B + \beta \quad \therefore B = \frac{\beta' - \beta}{\sqrt{\mu}}.$$

wherefore we have

$$x = a' - \frac{f}{\mu} + \beta t + \tfrac{1}{2} f t^2 + \left(a - a' + \frac{f}{\mu}\right) \cos\sqrt{\mu}\, t + \frac{(\beta' - \beta)}{\sqrt{\mu}} \sin\sqrt{\mu}\, t.$$





Let A be the initial pos^n of the particle. BC, BC' produced indefly. a section of the plate. AB ⊥ to CC'. ∴ AB = a.

Let BR = x. RR' = dx. ∴ $\pi(x+dx)^2 - \pi x^2 = 2\pi x\,dx$ nearly.

or $2\pi x\,dx \cdot k\rho$ = mass of the annulus of wh. RR is a sect^n.

The attraction of this upon A in the direct^n AB is $\frac{\cos RAB}{AR^2} \times 2\pi x\,dx k\rho$

$$= \frac{2\pi k\rho\, a\, x\,dx}{(a^2+x^2)^{\frac{3}{2}}} = dF$$

∴ the whole aff^n of the plate upon the particle in direct^n AB is

$$2\pi k\rho a \int_0^\infty \frac{x}{(a^2+x^2)^{\frac{3}{2}}} = 2\pi k\rho a \left\{ \frac{1}{(a^2+x^2)^{\frac{1}{2}}} + C \right\}.$$

Let x = 0 and this becomes. $\infty = 2\pi k\rho a \times 0 + C$

$$\therefore \quad C = 0.$$

Let x = a and we have $\int \frac{x}{x\,(a^2+x^2)^{\frac{3}{2}}} = \frac{1}{a}$.

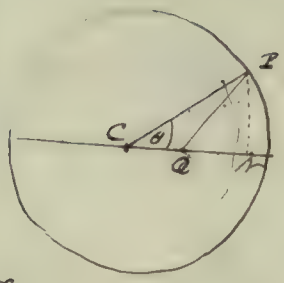
$$\therefore 2\pi k\rho a \int_0^\infty \frac{x}{(a^2+x^2)^{\frac{3}{2}}} = 2\pi k\rho = F = -\frac{d^2x}{t^2}.$$

Hence $2\frac{dx}{t}\frac{d^2x}{t^2} = -4\pi k\rho \cdot \frac{dx}{t}$

$$\left(\frac{dx}{t}\right)^2 = 4\pi k\rho (a-x) \quad \therefore \frac{dx}{t} = \frac{1}{2\sqrt{\pi k\rho}} \frac{1}{(a-x)^{\frac{1}{2}}}.$$

$$t = \frac{1}{(\pi k\rho)^{\frac{1}{2}}} (a-x)^{\frac{1}{2}}.$$ But when the particle arrives at the plate x = 0.

$$\therefore T = \frac{a^{\frac{1}{2}}}{(\pi\rho k)^{\frac{1}{2}}}.$$



$CQ = x \quad PQ = r \quad CP = a$

Repulsion of P on δr on Q $= \dfrac{\delta r}{PQ^2} = \dfrac{\delta r}{r^2}$

Force on Q in direction QC $= \dfrac{\delta r}{r^2} \cdot \dfrac{a - x}{r} = \dfrac{\delta r \cdot (a\cos\theta - x)}{(a^2 + x^2 - 2ax\cos\theta)^{\frac{3}{2}}}$

$$= \frac{a \cdot (a\cos\theta - x)\,\delta\theta}{(a^2 + x^2 - 2ax\cos\theta)^{\frac{3}{2}}}$$

$$= \delta\theta \cdot \frac{(a\cos\theta - x)}{a^2}\left\{1 + 3\frac{x}{a}\cos\theta\right\}$$

$$= \frac{\delta\theta}{a^2}\left\{a\cos\theta - x + 3x\cos^2\theta\right\}$$

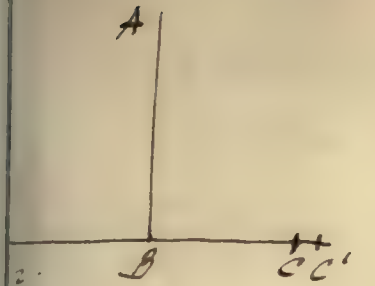
whole Repulsion $= \dfrac{2}{a^2}\int_0^{\pi}\left(a\cos\theta + \frac{3x}{2} + \frac{3}{2}x\cos 2\theta\right)$

$$= \frac{2}{a^2} \cdot \frac{x}{2} \cdot \pi = \frac{\pi \cdot x}{a^2}$$

$$\frac{d^2x}{dt^2} = -\frac{\pi k}{a^2} \cdot x$$

$$x = A\sin\left(\sqrt{\frac{\pi k}{a^2}} \cdot t + C\right)$$

Time of an oscillation $= \dfrac{2\pi}{\sqrt{\frac{\pi k}{a^2}}} = 2a\sqrt{\frac{\pi}{k\rho}}$



A the initial position of the particle
AB = a. ρ = density k = thickness of the plate.

Then the mass of a portion of the plate contained between radii BC. BC' at a distance. x and x+δx from each other

= 2πρk x δx.

The distance of which particle from A = $\sqrt{a^2+x^2}$.

∴ attract. of the circular ring in direction AB =

$$2\pi\rho k a \frac{x\,\delta x}{(a^2+x^2)^{\frac{3}{2}}}.$$

∴ whole attract of the plate in direct BA =

$$2\pi\rho k a \int_0^{x_0} \frac{x}{(a^2+x^2)^{\frac{3}{2}}} = B 2\pi\rho k.$$

$$\therefore \frac{d^2x}{dt} = +2\pi\rho k.$$

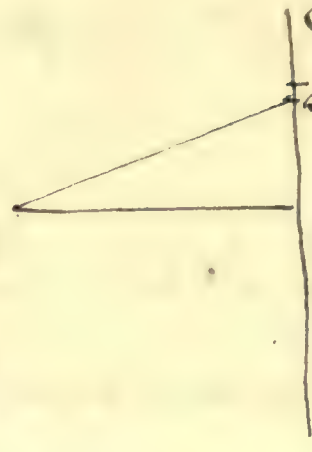
$$\frac{dx}{dt} = 2\pi\rho k t$$

$$x - a = \pi\rho k t^2. \quad \text{when } t=0 \; x=a.$$

$$\therefore t = \left(\frac{a}{\pi\rho k}\right)^{\frac{1}{2}}$$ for x=0 when the particle arrives at B.

for the time depends only on the absolute distance

CC' a section of the circular plate. B. its centre. AB ⊥ to CC'. A the material position of the particle. AB = a. BC = b.



Let Bx = x. xx' = δx. Then if ρ = density and k = thickness of the plate, the mass of the circular annulus whose radii are x and x + δx is $= k\rho\pi(x+\delta x)^2 - k\rho\pi x^2 = 2\pi x\delta x.k\rho$ nearly.

∴ the attn. of this annulus upon A in the direction AB is

$$2\pi k\rho.\frac{a\,x\,dx}{(a^2+x^2)^{\frac{3}{2}}}$$

and the whole attn. of the plate $= F = -\frac{d^2x}{t} = \int_0^b 2\pi k\rho a.\frac{x}{(a^2+x^2)^{\frac{3}{2}}}$.

$$\therefore +\frac{d^2x'}{t} = -2\pi k\rho x'\left(\frac{1}{(a^2+x^2)^{\frac{1}{2}}} + C\right) = -2\pi k\rho\left\{1 - \frac{x}{(b^2+x^2)}\right\}.$$

$$2\frac{d^2x}{t}\frac{dx}{t} = 4\pi k\rho\left\{\frac{x\frac{dx}{t}}{\sqrt{b^2+x^2}} - \frac{dx}{t}\right).$$

$$\therefore \left(\frac{dx}{t}\right)^2 = 4\pi k\rho\left\{\left(\sqrt{b^2+x^2} - x\right) + C\right).$$

when $\left(\frac{dx}{t}\right)^2 = 0$ $x = a$

$$\therefore\ 0 = 4\pi k\rho\left\{\sqrt{b^2+a^2} - a\right| + C\right).$$

$$\therefore\ C = 4\pi k\rho\left(a - \sqrt{b^2+a^2}\right).$$

and $\left(\frac{dx}{t}\right)^2 = v^2 = 4\pi k\rho\left\{\sqrt{b^2+x^2} - x + a - \sqrt{b^2+a^2}\right)$

at B $x = 0$

$$\therefore V^2 = 4\pi k\rho\left\{a + b - \sqrt{a^2+b^2}\right).$$

Let a = height from wh. the body falls.

v = velocity acquired by fall.

$\therefore v^2 = 2ga$

e^2v^2 = (velocity)^2 with wh. it rebounds $\therefore ae^2$ = height it attains

Similarly ae^4 = height wh. it attains after second rebound.

ae^6 = ——————— third ———

$\therefore$ whole space described $= a\{1+2e^2+2e^4+2e^6+\ldots\}$.

$= a\left\{\frac{1+e^2}{1-e^2}\right\}$.

Let s & s' be the heights from which the balls fall. s' being greater than s.

Then $\sqrt{2gs}$ and $\sqrt{2gs'}$ are the velocities acquired by falling.

Also $t = \frac{v}{f}$ $\therefore \sqrt{\frac{2s}{g}}$ and $\sqrt{\frac{2s'}{g}}$ are the times of descent.

$\therefore \sqrt{\frac{2}{g}}\left(\sqrt{s'} - \sqrt{s}\right)$ = time elapsing between the moments of striking the plane.

Let x = time the second ball requires to overtake the first ball after it (the second ball) strikes the plane.

Then $x \cdot \sqrt{2gs'} = \left\{x + \sqrt{\frac{2}{g}}\left(\sqrt{s'} - \sqrt{s}\right)\right\}\sqrt{2gs}$

$\therefore x\left(\sqrt{s'} - \sqrt{s}\right) = \sqrt{\frac{2}{g}}\cdot\left(\sqrt{s'} - \sqrt{s}\right)\sqrt{s}$

$\therefore x = \sqrt{\frac{2s}{g}}$

Also $x \times \sqrt{2gs'}$ = space req^{d}.

$\therefore \sqrt{\frac{2s}{g}} \times \sqrt{2gs'} = 2\sqrt{ss'}$ = space req^{d}.

let AB = a. v = velocity of proj. from B.

$\therefore s' = \frac{v^2}{2g}$ = greatest elevat. attained by B.

let t = time of B's attaining this elevation.

s = dist. from A at wh. collision takes place.

Then $s = \frac{1}{2} g (t+t')^2$, where t' = time that elapses after B has attained its greatest elevat. before collision takes place.

also $s - a + \frac{v^2}{2g} = \frac{1}{2} s t'^2$

$\therefore \frac{g}{2} \{ t^2 + 2tt' + t'^2 \} = \frac{g}{2} t'^2 + a - \frac{v^2}{2g}$.

$g t^2 + 2g t t' = 2a - \frac{v^2}{g}$

$t' = \frac{2a - \frac{v^2}{g} - g t^2}{2g t} = \frac{2a - 2\frac{v^2}{g}}{2g t}$ (for $t = \frac{v}{g}$)

$= \frac{ag - v^2}{gv}$

But $t = \frac{v}{g} = \frac{v^2}{gv}$

$\therefore t + t' = \frac{ag - v^2 + v^2}{gv} = \frac{a}{v}$. = right time

also $s = \frac{1}{2} f t^2 \therefore s = \frac{g}{2} \cdot \frac{a^2}{v^2}$ = space req.

Bady

adopting the notation the $=^n$ of motion will be

$$\frac{d^2x}{t} = +\frac{\mu}{x^3} - \frac{h}{x^2}\left(\frac{dx}{t}\right)^2.$$

$$2\frac{dx}{t}\frac{d^2x}{t} = \frac{2\mu}{x^2}\frac{dx}{t} - \frac{2h}{x^2}\cdot\left(\frac{dx}{t}\right)^3. \quad \text{let } \left(\frac{dx}{t}\right)^2 = w. \ \frac{1}{x} = z.$$

$$\therefore d_t w = -2\mu d_t z^2 + \frac{2h}{}\left(\frac{dx}{t}\right)^2 \cdot d_t z$$

$$dw = -2\mu z\, dz + 2hw\, dz.$$

$$dw - 2hw\, dz = -2\mu z\, dz.$$

$$d\left(we^{-2hz}\right) = -2\mu z e^{-2hz}\, dz.$$

$$\therefore e^{-2hz} w = -2\mu \int z\, e^{-2hz} = -2\mu\left\{\frac{z}{-2h}e^{-2hz} + \frac{1}{4h^2}e^{-2hz}\right\} = \frac{\mu}{2h^2}\left\{2hz+1\right\}e^{-2hz}$$

$$\text{or } e^{-\frac{2h}{x}} v^2 = \frac{\mu}{2h^2}\left(\frac{2hx+x}{x}\right)e^{-\frac{2h}{x}} + C.$$

$$\text{or } e^{\frac{2h}{x}} v^2 = \frac{\mu}{2h^2}\left(\frac{2hx-x}{x}\right)e^{\frac{2h'}{x}} + C. \quad (A)$$

$$e^{\frac{2h}{a}}\beta^2 = \frac{\mu}{2h^2}\left(\frac{2ha-a}{a}\right)e^{\frac{2h'}{a}} + C.$$

$$\therefore e^{\frac{2h}{x}} v^2 - e^{\frac{2h}{a}}\beta^2 = \frac{\mu}{2h^2}\left\{\frac{a-2h'}{a}e^{\frac{2h'}{a}} - \left(\frac{x-2h}{x}\right)e^{\frac{2h'}{x}}\right\}$$

Multiply by $e^{-\frac{2h}{ax}}$.

$$e^{-\frac{2h}{a}} v^2 - e^{-\frac{2h}{x}}\beta^2 = \frac{\mu}{2h^2}\left\{\frac{a-2h'}{a}e^{-\frac{2h'}{x}} - \left(\frac{a-2h'}{x}\right)e^{-\frac{2h}{a}}\right\}.$$

(A) this $=^n$ is obtained from the $=^n$. $\frac{d^2x}{t} = -\frac{\mu}{x^3} + \frac{h}{x^2}\left(\frac{dx}{t}\right)^2$.

(4)

like $\frac{dv}{dt} = \mp 2k\sqrt{v}$.

$\therefore \frac{dt}{dv} = \mp \frac{1}{k\sqrt{v}}$

$\therefore t = \pm \frac{2\sqrt{v}}{k} + C$.

$\theta = \pm \frac{2\sqrt{\beta}}{k} + C$

$\therefore t = \pm \frac{2\sqrt{\beta}}{k} - \frac{2\sqrt{v}}{k}$. $\therefore \sqrt{v} = \left(\pm\frac{k}{2} - \sqrt{\beta}\right)$

when the body comes at rest $v = 0$

$\therefore t = \pm \frac{2\sqrt{\beta}}{k}$.

To find the space described in any time t.

$v\,dv = -k\sqrt{v}$.

$\int_\beta^v v^{\frac{1}{2}} = -\int k$

$\therefore v^{\frac{3}{2}} = -\frac{k}{2}x + C$.

when $x = 0$ $v = \beta$

$\therefore \beta^{\frac{3}{2}} = C$.

$\therefore v^{\frac{3}{2}} = -\frac{k}{2}x + \beta^{\frac{3}{2}}$.

$\left(\pm\frac{k}{2} - \sqrt{\beta}\right)^{\frac{3}{2}} = -\frac{k}{2}x + \beta^{\frac{3}{2}}$.

$\therefore x = \frac{1}{k}\left\{\beta^{\frac{3}{2}} - \left(\pm\frac{k}{2} - \sqrt{\beta}\right)^{\frac{3}{2}}\right\}$.

$$\frac{dv}{dt} = g - kv^2.$$

$$dt = \frac{1}{g - kv^2} = \frac{1}{k\left(\frac{g}{k} - v^2\right)}.$$

$$t = \frac{1}{2\sqrt{kg}} \cdot \log_e \left| \frac{\sqrt{\frac{g}{k}} + v}{\sqrt{\frac{g}{k}} - v} \right| + C.$$

$t = 0 \quad v = 0.$

$$\therefore 0 = \frac{1}{2\sqrt{kg}} \cdot \log 1 + C.$$

$$\therefore t = \frac{1}{2\sqrt{kg}} \cdot \log \frac{\sqrt{\frac{g}{k}} + v}{\sqrt{\frac{g}{k}} - v}.$$

$$\therefore v = \sqrt{\frac{g}{k}} \cdot \frac{e^{2t\sqrt{kg}} - 1}{e^{2t\sqrt{kg}} + 1}$$

$$\frac{dv'}{dt'} = -g - kv'^2.$$

$$dt' = \frac{dv'}{-g - kv'^2} = -\frac{1}{k} \cdot \frac{1}{\frac{g}{k} + v'^2}$$

$$\therefore t' = -\frac{1}{\sqrt{gk}} \tan^{-1} \frac{v'\sqrt{k}}{\sqrt{g}} + C$$

$$t' = -\frac{1}{\sqrt{gk}} \tan^{-1} \frac{v'\sqrt{k}}{\sqrt{g}} + C.$$

$t' = 0 \quad v' = \sqrt{\frac{g}{k}} \cdot \frac{e^{2t\sqrt{kg}} - 1}{e^{2t\sqrt{kg}} + 1}$

$$t' = \frac{1}{\sqrt{gk}} \left\{ \tan^{-1} \frac{e^{2t\sqrt{kg}} - 1}{e^{2t\sqrt{kg}} + 1} - \tan^{-1} \frac{v'\sqrt{k}}{\sqrt{g}} \right\}.$$

$t = t' = T \quad v' = 0.$

$$T\sqrt{gk} = \tan^{-1} \frac{e^{2t\sqrt{kg}} - 1}{e^{2t\sqrt{kg}} + 1}$$

$$\frac{1}{n}\,T.\sqrt{gh} = \frac{\xi^{2\frac{t}{n}\sqrt{hg}} - 1}{\xi^{2\frac{t}{n}\sqrt{hg}} + 1}.$$

$$\xi^{2\frac{t}{n}\sqrt{hg}} + 1 \,.\, \frac{1}{n}\,T.\sqrt{gh} = \xi^{2\frac{t}{n}\sqrt{hg}} - 1.$$

$$\xi^{2\frac{t}{n}\sqrt{hg}} \left\{ \frac{1}{n}\,T.\sqrt{gh} - 1 \right\} = \left(1 - \frac{1}{n}\,T.\sqrt{gh} \right) \xi^{2\frac{t}{n}\sqrt{hg}}$$

$$= \frac{1}{n}\,T.\sqrt{gh} + 1 = 1 - \frac{1}{n}\,T.\sqrt{gh}.$$

$$\frac{1 + \frac{1}{n}\,T.\sqrt{gh}}{1 - \frac{1}{n}\,T.\sqrt{gh}} = \xi^{2\frac{t}{\sqrt{\frac{h}{g}}}}.$$

$$\frac{1}{n}\left(\frac{\pi}{4} + T.\sqrt{gh} \right) = \xi^{2\frac{t}{\sqrt{\frac{h}{g}}}}.$$

$$\log \frac{1}{n}\left(\frac{\pi}{4} + T.\sqrt{gh} \right) = 2\frac{t}{\sqrt{\frac{h}{g}}}.$$

x = no of motors until t.

$$\frac{d^2x}{dt^2} = -k.$$

$$\therefore \frac{dx}{dt} = -kt + C.$$

when $t=0$. $\frac{dx}{dt} = 12000$ miles,

$$\therefore \frac{dx}{dt} = 12000 - kt. = 6000 \text{ by question.}$$

$$\therefore k = \frac{6000}{t}. \quad (1)$$

Integrate again and obtain.

$$x = 12000t - \frac{k}{2}.t^2$$

$$= 9000t \text{ by substitution.}$$

But $x = 3$ by question.

$$\therefore t = \frac{1}{3000}.$$

1 } =0.

$$\frac{dv}{t} = -f + \frac{kv^2}{x}.$$

$$d_x(v^2) = -2f\,dx + \frac{2kv^2}{x}\,dx$$

$$d(v^2) - \frac{2kv^2}{x}\cdot dx = -2f\,dx.$$

$$d.\{v^2 x^{-2k}\} = -2f x^{-2k}\cdot dx$$

$$v^2 x^{-2k} = \frac{-2f}{1-2k}\cdot x^{1-2k} + C.$$

when $v = v \quad x = a$

$$\therefore C = -\frac{2f}{1-2k}\,a^{1-2k} + C.$$

$$\therefore v^2 x^{-2k} = \frac{2f}{1-2k}\{a^{1-2k} - x^{1-2k}\}$$

$$v^2 = \frac{2f}{1-2k}x^{2k}\{a^{1-2k} - x^{1-2k}\}$$

$$2v\,dv_x = \frac{2f}{1-2k}\{2kx^{2k-1}\{a^{1-2k} - x^{1-2k}\} + 0\{(1-2k)a\}\} = 0.$$

for a max $\quad -2k-1+2k + 2k\cdot x'^{-(1-2k)}\,a^{1-2k} = 0.$

$$\frac{2k}{(x')^{1-2k}} = \frac{1}{(a)^{1-2k}}.$$

$$\left(\frac{x'}{a}\right)^{1-2k} = 2k^{\frac{1}{1-2k}}$$

$$\therefore x' = a.(2k)^{\frac{1}{1-2k}}$$

But $x+x' = a = v + \frac{v'-\beta}{h} = -\frac{\beta v}{g}$ from (3).

$\therefore ag = -\beta v$ or $v = -\frac{ag}{\beta}$.

$$t = \frac{1}{h}\log\left\{\frac{g-vh}{g}\right\} = \frac{1}{h}\log\left\{\frac{g+\frac{hag}{\beta}}{g}\right\}$$

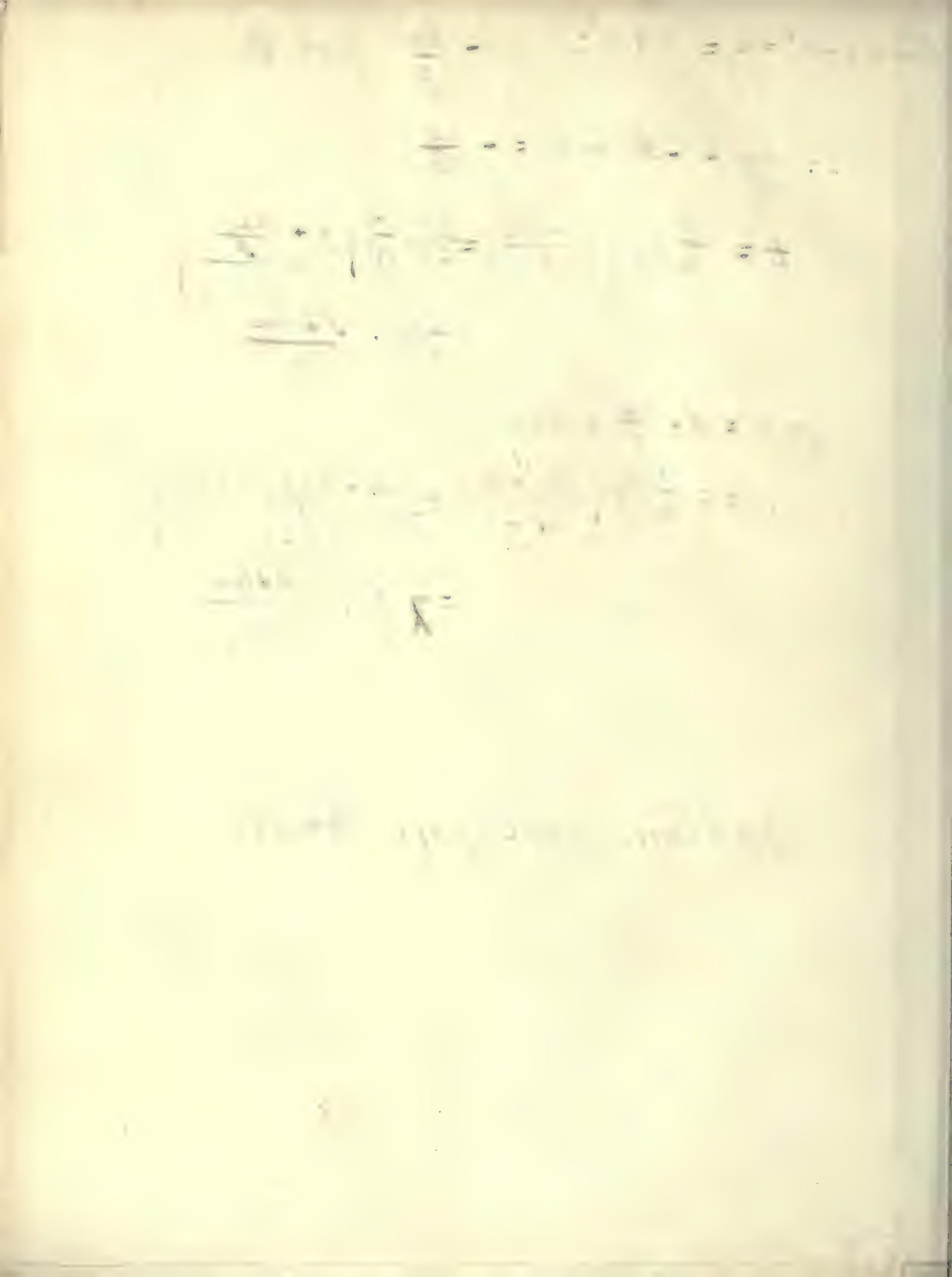
$$= \frac{1}{h}\log\cdot\frac{\beta+ha}{\beta}$$

or $v' = \beta + \frac{ag}{\beta} + ha$.

$$\therefore t = \frac{1}{h}\log\left\{\frac{hv'+g}{h\beta+g}\right\} = \frac{1}{h}\log\left\{\frac{h\beta + \frac{hag}{\beta} + h^2a + g}{h\beta+g}\right\}$$

$$= \frac{1}{h}\log\cdot\left(\frac{\beta+ha}{\beta}\right)$$

Walton. page 174. Ex 12.



A

Let v be the velocity of the falling particle at time t.

$$\therefore \frac{dv}{dt} = -g + kv$$

$$\therefore dt = \frac{1}{kv - g} \quad \therefore t = \frac{1}{k} \cdot \log.\left(v - \frac{g}{k}\right) + C.$$

when $t = 0$ $v = 0$ $\therefore t = \frac{1}{k} \cdot \log.\left(\frac{g - vk}{g}\right)$. (1)

B

Similarly for the projected particle.

$$\frac{dv'}{dt'} = g + kv.$$

$$\therefore t' = \frac{1}{k} \log.\left(v' + \frac{g}{k}\right) + C'$$

when $t' = 0$ $v' = \beta$.

$$\therefore t' = \frac{1}{k} \cdot \log.\left\{\frac{kv' + g}{k\beta + g}\right\}. \quad \text{— (2)}$$

If the time $t = t'$

$$\therefore \frac{g - vk}{g} = \frac{kv' + g}{k\beta + g}.$$

$$gk\beta + g^2 - vk^2\beta - vkg = gkv' + g^2$$

$$v' = \beta - v - \frac{k\beta v}{g}. \quad (3)$$

Again $v\,dv = -g + kv$.

$$du = -\frac{g}{v} + k.$$

$$dx = \frac{v}{kv - g} = \frac{1}{k} + \frac{g}{k^2} \cdot \frac{1}{v - \frac{g}{k}}$$

$$x = \frac{v}{k} + \frac{g}{k^2} \log. \frac{g - vk}{g} = \frac{v}{k} + g\frac{t}{k}.$$

$$v'\frac{dv'}{dx'} = g + kv$$

$$u' = \frac{v - \beta}{k} - \frac{g}{k^2} \cdot \log.\left(\frac{kv' + g}{k\beta + g}\right) = \frac{v'}{k} - g\frac{t}{k}.$$

$$x + x' = a = \left(\frac{v + v' - \beta}{k}\right) = \frac{g\beta - k\beta^2}{gk} = \frac{\beta}{g}\left(g - k\frac{\beta}{k}\right),$$

Let Aa = direct. of prop. $\therefore \angle aAC = \alpha$.

$\angle BAC = \theta$

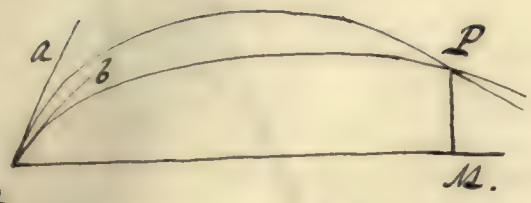
$AB = a$.

Then $\frac{a}{n}$ = velocity of prop.

also $a \cos\theta$ = whole horiz. space described

$\frac{a}{n}\cos\alpha$ = const horiz. velocity

$\therefore \dfrac{a\cos\theta}{\frac{a}{n}\cos\alpha} = n\dfrac{\cos\theta}{\cos\alpha}$ = whole time of flight.



A. the point of proj. $\angle a = \alpha$ $\angle b = \beta$ the direcn of proj.
T the reqd time, u u' = velocity of proj.
$AM = x$. $ML = y$ the co-ordts of the pt. common to both paths.

The eqn to the paths are $y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$

$y = x\tan\beta - \frac{gx^2}{2u'^2\cos^2\beta}$

$\therefore \tan\alpha - \tan\beta = \frac{gx}{2}\left\{\frac{1}{u^2\cos^2\alpha} - \frac{1}{u'^2\cos^2\beta}\right\}$.

$\frac{2}{g}\left(\sin\alpha\cos\beta - \cos\alpha\sin\beta\right) = \left(\frac{u'^2\cos^2\beta - u^2\cos^2\alpha}{u^2u'^2\cos\alpha\cos\beta}\right)x$.

$\therefore x = AM = \frac{2}{g}\cdot\frac{\sin(\alpha\sim\beta)\,u^2u'^2\cos\alpha\cos\beta}{u'^2\cos^2\beta - u^2\cos^2\alpha}$.

also the comb. velocity of $A = u\cos\alpha$.

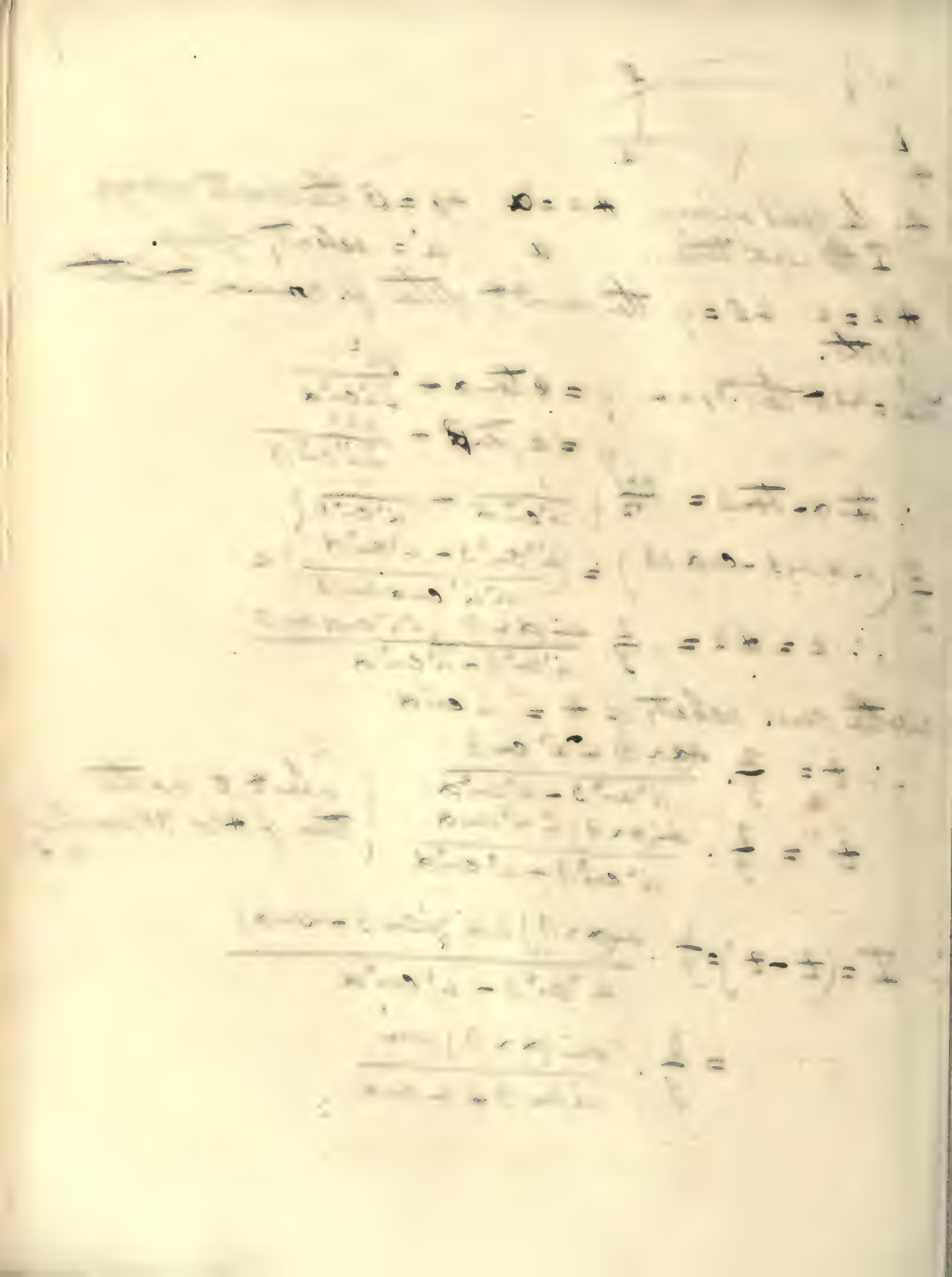
$\therefore t = \frac{2}{g}\cdot\frac{\sin(\alpha\sim\beta)\,u^2u'^2\cos\beta}{u'^2\cos^2\beta - u^2\cos^2\alpha}$.

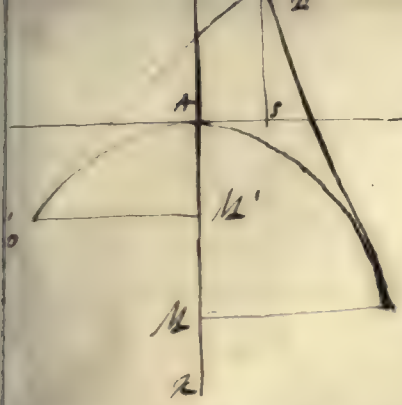
$t' = \frac{2}{g}\cdot\frac{\sin(\alpha\sim\beta)\,u'u^2\cos\alpha}{u'^2\cos^2\beta - u^2\cos^2\alpha}$

} where t, t' are the times of A & B to reach L.

$\therefore T = (t - t') = \frac{2}{g}\cdot\frac{\sin(\alpha\sim\beta)\,uu'\{u'\cos\beta - u\cos\alpha\}}{u'^2\cos^2\beta - u^2\cos^2\alpha}$

$= \frac{2}{g}\cdot\frac{\sin(\alpha\sim\beta)\,uu'}{u'\cos\beta + u\cos\alpha}$.





A the vertex of the P. Ax. Ay co-ordt axes. P'AP the arc described. v_1 v_2 v_3 the velocities at P. A P respectively. TS a vertical line // to Ay axis of yx. $\angle PTS = \theta_3$ $\angle P'TS = \theta_1$

The horizl. velocities at P' P are. $v_1 \sin\theta_1$, $v_3 \sin\theta_3$

$\therefore v_1 \sin\theta_1 = v_2 = v_3 \sin\theta_3$ for the horizl. vel. is const.

The vertical velocities are $v_1 \cos\theta_1$ $v_3 \cos\theta_3$ —

$\therefore v_1\cos\theta_1$, $v_3 \cos\theta_3$ are the velocities acquired by particles falling thro. the spaces AM' AM respectively.

$$\therefore AM' = \frac{v_1^2 \cos^2\theta_1}{2g} \quad , \quad AM = \frac{v_3^2 \cos^2\theta_3}{2g}$$

also if t t' represent the times of describing the arcs AP. AP'
$\therefore$ the times of falling thro' AM' AM. where

$$s = \tfrac{1}{2} f t''^2 \quad \therefore t''^2 = \frac{2s}{f}$$

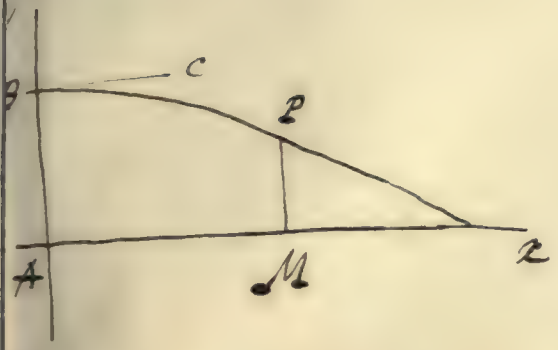
$$\therefore t = v_3 \frac{\cos\theta_3}{g} \quad , \quad t' = \frac{v_1 \cos\theta_1}{g}$$

$$\therefore t + t' = \frac{v_3\sqrt{1-\sin^2\theta_3} + v_1\sqrt{1-\sin^2\theta_1}}{g}$$

$$= v_3 \sqrt{\frac{}{g}}$$

$$t + t' = v_1 v_3 \frac{v_2^2}{v_1^2} \frac{\cos\theta_3}{v_2 g} + \frac{v_1 v_3 \frac{v^2}{v_3}\cos\theta_1}{v_2 g}$$

$$= \frac{v_1 v_3}{v_2 g}\left\{\sin\theta_1\cos\theta_3 + \cos\theta_1\sin\theta_3\right\} = \frac{v_1 v_3 \sin P\Pi P'}{v_2 g}$$



A the origin of co-ords. Ax. Ay the axes of x. y. BC // to Ax the dirct. of force. B the velocity of proj. μ the force of grav. attract.. Ax = x. xP = y

co-ords of the particle at any time t. BA = b.

The eqns of mot. are $\frac{d^2y}{dt^2} = -\mu y$ $\frac{d^2x}{dt^2} = 0$

$$\left(\frac{dy}{dt}\right)^2 = -\mu y^2 + C$$

also $\left(\frac{dy}{dt}\right) = 0$ when $y = a$

$$\therefore \left(\frac{dy}{dt}\right)^2 = \mu a^2 - y^2$$

$$\frac{dy}{dt} = \frac{1}{\sqrt{\mu}} \frac{1}{\sqrt{a^2 - y^2}}$$

$$t = \frac{1}{\sqrt{\mu}} \cos^{-1} \frac{y}{a}$$

$$\therefore y = a \cos \sqrt{\mu}\, t. \quad (1)$$

also $\frac{d^2x}{dt^2} = 0 \quad \therefore \frac{dx}{dt} = C = b$

$$\therefore \frac{dt}{dx} = \frac{1}{b} \text{ or } t = \frac{x}{b} \quad \therefore x = bt.$$

wherefore $x = bt$. $y = a \cos \sqrt{\mu}\, t.$

or $y = a \cos \frac{x}{b} \sqrt{\mu}.$

A the point of prop. A y the dist. of prop.
AG = a. the initial posn. of the body u =
const. velocity of cent. of fce. v = velocity
of prop. R = posn. of the C of fce at time t. P. the posn. of
the particle at the same time. AM = x. MP = y coordts of P.

the eqns of motn. are $\frac{d^2y}{dt^2} = -\mu y$. (1) $\frac{d^2x}{dt^2} = \mu MQ$

$= \mu(a+ut-x)$ (2)

from (1) $\left(\frac{dy}{dt}\right)^2 = C - \mu y^2$

$v^2 = C \quad \therefore \left(\frac{dy}{dt}\right)^2 = v^2 - \mu y^2$ (3)

when the particle moves to the axis of x $\left(\frac{dy}{dt}\right)^2 = 0$

$\therefore y = \frac{v}{\sqrt{\mu}}$.

from (2) $\frac{d^2x}{dt^2} = \mu(a+ut-x)$

$\frac{d^2x}{dt^2} + \mu x = \mu(a+ut)$.

$x = a + ut + A\cos\sqrt{\mu}t + B\sin\sqrt{\mu}t$.

$\frac{dx}{dt} = u - \sqrt{\mu}A\sin\sqrt{\mu}t + \sqrt{\mu}B\cos\sqrt{\mu}t$.

$0 = a + A \quad \therefore A = -a$.

$0 = u + \sqrt{\mu}B \quad \therefore B = -\frac{u}{\sqrt{\mu}}$

$x = a + ut - a\cos\sqrt{\mu}t - \frac{u}{\sqrt{\mu}}\sin\sqrt{\mu}t$. (2)

Also from (1) $t = \frac{1}{\sqrt{\mu}}\cdot\sin^{-1}\frac{\sqrt{\mu}.y}{u} = \frac{1}{\sqrt{\mu}}\frac{\pi}{2}$ (3) when the

$y = \frac{u}{\sqrt{\mu}}\sin\sqrt{\mu}t$ limits $y = 0$, $y = \frac{u}{\sqrt{\mu}}$;

substitute the value of t in (3) in (2) and we have

$x = a + \frac{u\pi}{\sqrt{\mu}.2} - \frac{u}{\sqrt{\mu}} = a + \frac{u}{2}\left\{\frac{\pi}{\sqrt{\mu}} - \frac{2u}{\sqrt{\mu}}\right\} = a + \frac{u}{2\sqrt{\mu}}\{\pi - 2\}$

$$r = a\left(\sec\frac{\theta}{n}\right)^n$$

$$u = \frac{\cos\left(\frac{\theta}{n}\right)^n}{a}$$

$$\frac{du}{u} = \frac{1}{a}\cdot n\cdot\log\cos\left(\frac{\theta}{n}\right)\frac{\sin\frac{\theta}{n}}{\cos\frac{\theta}{n}} = \tan\frac{\theta}{n} = -\frac{1}{a}\tan\frac{\theta}{n}$$

$$\therefore \frac{du}{d\theta} = -\frac{1}{a}\cdot u\tan\frac{\theta}{n}$$

$$\frac{d^2u}{d\theta^2} = -\frac{1}{a}\left\{u\sec^2\frac{\theta}{n} + \tan^2\frac{\theta}{n}\right\}$$

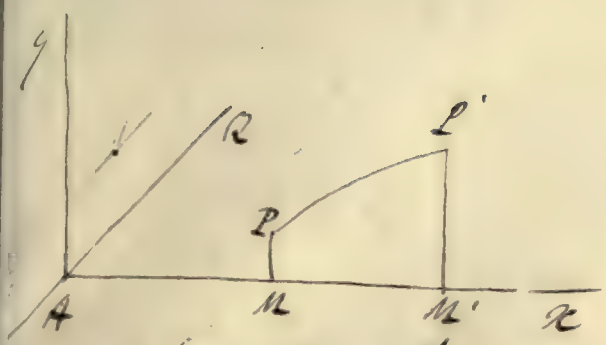
$$= -\frac{1}{a}\cdot u\left\{\tan^2\left(\frac{\theta}{n}\right)^n + \sec^2\left(\frac{\theta}{n}\right)^n - 1\right\}$$

$$\therefore \frac{d^2u}{d\theta^2} + u = -\frac{1}{a}\left\{u+1\right\}\left(\sec^2\frac{\theta}{n}\right) = -\frac{1}{a}$$

$$\frac{du}{d\theta} = \frac{1}{a}\cdot a\left(\frac{\theta}{n}\right)^n \qquad F = d^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$\frac{d^2u}{d\theta^2} = \frac{1}{a}\cdot\cos\left(\frac{\theta}{n}\right)^n \qquad = -\frac{1}{a}\left(u\tan^2\frac{\theta}{n} + u + \tan^2\left(\frac{\theta}{n}\right)\right.$$

$$\therefore \frac{d^2u}{d\theta^2} + u = \frac{1}{a}\cos\left(\frac{\theta}{n}\right)^n \qquad \frac{d^2u}{d\theta^2} + u = -\frac{1}{a}(u+1)\tan^2\left(\frac{\theta}{n}\right.$$



A the cent. of the repulsive fce the origin co-ordy.
P the mobile point of the body, P' its posn after any time t. R the direct. in wh. the const. fce acts.

The eqns of motn are. $\frac{d^2y}{t} = \mu\left(y - \frac{f}{\mu\sqrt{2}}\right)$ (1)

$$\frac{d^2x}{t} = \mu\left(x - \frac{f}{\mu\sqrt{2}}\right) \quad (2)$$

from the (1) $\frac{d^2}{t^2}\left(y - \frac{f}{\mu\sqrt{2}}\right) - \mu\left(y - \frac{f}{\mu\sqrt{2}}\right) = 0$

$$y = A\cos\sqrt{\mu}\,t + B\sin\sqrt{\mu}\,t + \frac{f}{\mu\sqrt{2}}.$$

when $y = b$ $t = 0$ $\therefore b + \frac{f}{\mu\sqrt{2}} = A.$

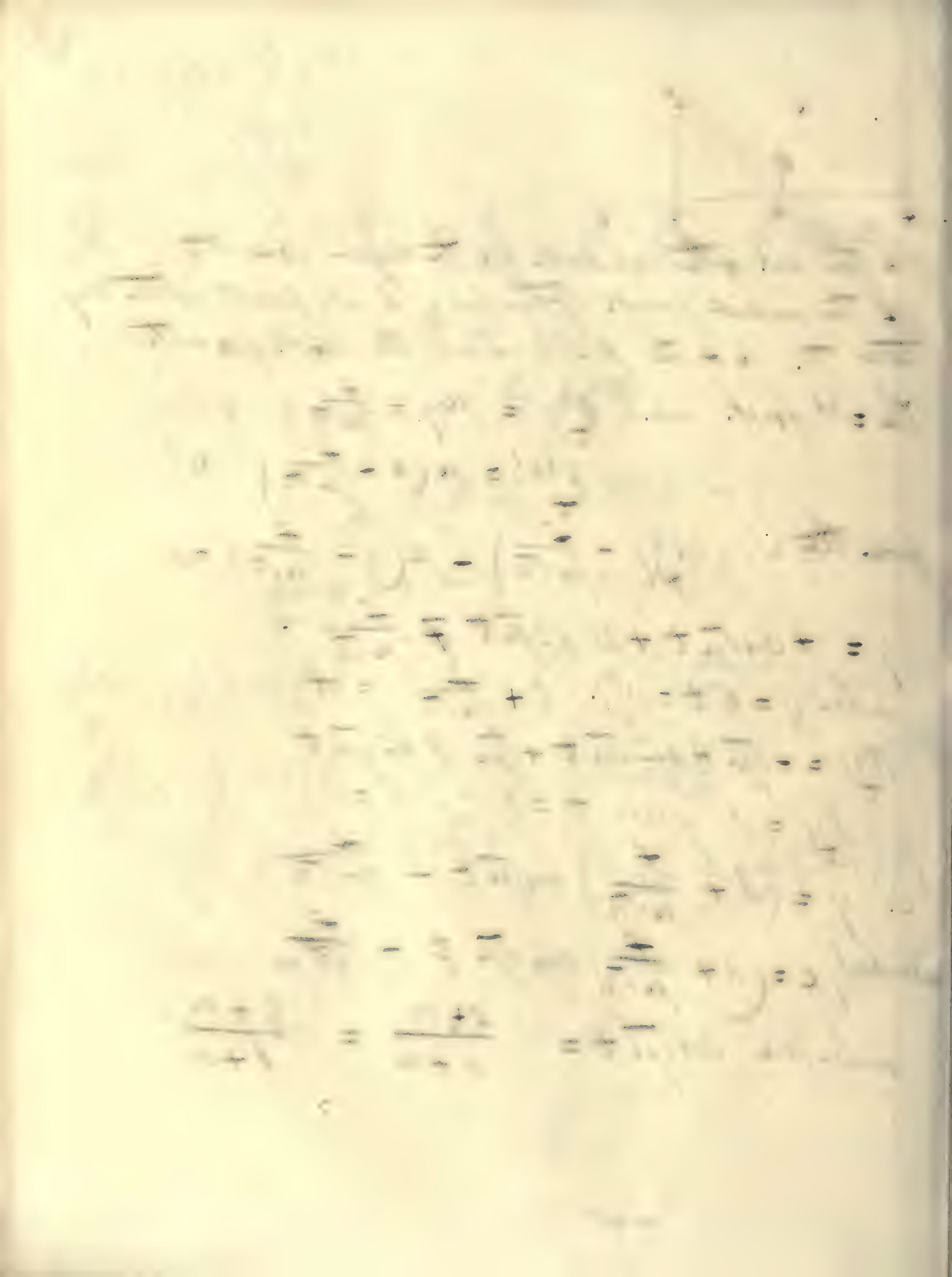
$$\frac{dy}{t} = -\sqrt{\mu}\,A\sin\sqrt{\mu}\,t + \sqrt{\mu}\,B\sin\sqrt{\mu}\,t$$

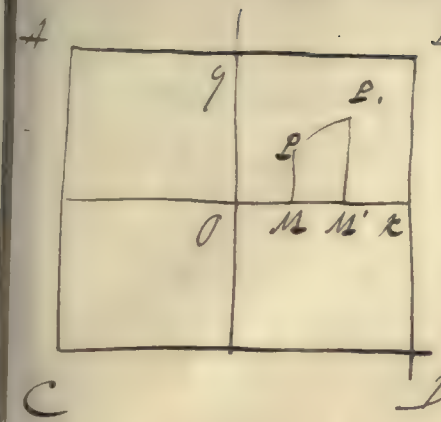
$\frac{dy}{t} = 0$ when $t = 0$ $\therefore B = 0$

$$\therefore y = \left(b + \frac{f}{\mu\sqrt{2}}\right)\cos\sqrt{\mu}\,t - \frac{f}{\mu\sqrt{2}}$$

Similarly $x = \left(a + \frac{f}{\mu\sqrt{2}}\right)\cos\sqrt{\mu}\,t - \frac{f}{\sqrt{2}.\mu}$

wherefore $\cos\sqrt{\mu}\,t = \frac{x+m}{a+m} = \frac{y+m}{b+m}.$





Let O be the origin of co-ordts. &c.
OM' ML. the co-ordts of the particle at the expiration of the time t. L its initial position.

The eqns of motn are $\frac{d^2y}{dt^2} = -2\mu(b+y) + 2\mu(b-y)$

$= -4\mu y$

$\frac{d^2x}{dt^2} = -4\mu x$ } (1)

$2\frac{dy}{dt}\frac{d^2y}{dt^2} = -8\mu y \frac{dy}{dt}$ $\therefore \left(\frac{dy}{dt}\right)^2 = -4\mu y^2 + C$.

when $\frac{dy}{dt} = 2m$ $y = b$ $\therefore \left(\frac{dy}{dt}\right)^2 = 4m^2 + 4\mu(b^2 - y^2)$

from wh. it is easily seen that at $y = \frac{1}{2\sqrt{\mu}} \frac{1}{\sqrt{m^2 + b^2\mu} - y^2}$

$\therefore t = \frac{1}{2\sqrt{\mu}} \left\{ \sin^{-1} \frac{\sqrt{\mu}\, y}{\sqrt{m^2 + b^2\mu}} + C \right.$

When $t = 0$ $y = b$

$\therefore 0 = \frac{1}{2\sqrt{\mu}} \sin^{-1} \frac{\sqrt{\mu}\, b}{\sqrt{m^2 + b^2\mu}} + C$

$\therefore 2t\sqrt{\mu} = \sin^{-1} \frac{\sqrt{\mu}\, y}{\sqrt{m^2 + b^2\mu}} - \sin^{-1} \frac{\sqrt{\mu}\, b}{\sqrt{m^2 + b^2\mu}}$.

Similarly $2t\sqrt{\mu} = \sin^{-1} \frac{\sqrt{\mu}\, x}{\sqrt{m^2 + a^2\mu}} - \sin^{-1} \frac{\sqrt{\mu}\, a}{\sqrt{m^2 + a^2\mu}}$

Therefore $\sin^{-1} \frac{\sqrt{\mu}\, y}{\sqrt{m^2 + b^2\mu}} - \sin^{-1} \frac{\sqrt{\mu}\, b}{\sqrt{m^2 + b^2\mu}} = \sin^{-1} \frac{\sqrt{\mu}\, x}{\sqrt{m^2 + a^2\mu}} - \sin^{-1} \frac{\sqrt{\mu}\, a}{\sqrt{m^2 + a^2\mu}}$

the attraction of gravity acting on both particles alike may be neglected in the question.

Let at the moment the projected particle attains its highest elevation its velocity is $v \cos\alpha$ in a horizontal direction.

The relative velocity of the radius drawn to the same direction will be v.

$\therefore$ the horizontal distance at the end of the time $\frac{1}{2}t$ $= vt \cos\alpha$.

relative vertical distance $\frac{1}{2}vt$.

$\therefore$ the whole distance $= \sqrt{\frac{1}{4}v^2t^2 + v^2t^2\cos^2\alpha}$

$= \frac{1}{2}vt\sqrt{1+4\cos^2\alpha}$.

α = the point of projection.

$\angle BAO = \alpha$ = the angle of projection

$\angle CAO = i$ = the inclination of the plane to the horizon.

Let x = the distance of the point from the plane at time t.

v = the velocity of the plane at that time

$\therefore x = vt\sin(\alpha - i) - \frac{1}{2}gt^2\cos i$

$v' = v\sin(\alpha - i) - gt\cos i$

also $x = v$ $\therefore v\sin(\alpha - i) = \frac{1}{2}gt\cos i$

$\therefore v' = \frac{1}{2}gt\cos i - gt\cos i = -\frac{1}{2}gt\cos i = -$ the velocity of A

$\therefore$ the velocity of the plane at A and is equal.

also the velocity, v_1, of the plane $= v\cos(\alpha - i) - gt\sin i$.

Now the time of flight $= \frac{2v\sin(\alpha - i)}{g\cos i}$

$\therefore v_1 \cos\alpha_1 = v\cos(\alpha - i) - \frac{2v\sin(\alpha - i)\sin i}{\cos i}$

$\therefore v/\alpha_1$ = tan of the angle of incidence at α.

$$\tan\alpha_1 = \frac{v\sin(\alpha - i)\cos i}{v\cos(\alpha - i)\cos i - 2v\sin(\alpha - i)\sin i}$$

$$= \frac{\tan(\alpha - i)}{1 - 2\tan(\alpha - i)\tan i}$$

also $\tan\beta_2 = \dfrac{e\tan(\alpha - i)}{1 - 2\tan(\alpha - i)\tan i}$

similarly from the second flight we have.

vertical velocity at 2' = vertical velocity of reflection

at 2 = $e v \sin(\alpha - \iota)$.

and the velocity at 2' of the plane =

$$\frac{v\cos(\alpha-\iota)\cos\iota - 2v\sin(\alpha-\iota)\sin\iota}{\cos\iota} - gt'\sin\iota'$$

but $t' = \frac{2ve\sin(\alpha-\iota)}{g\cos\iota'}$

$$\therefore \text{vel at 2'} = \frac{v\cos(\alpha-\iota)\cos\iota' - 2v\sin(\alpha-\iota)\sin\iota - 2ve\sin(\alpha-\iota)\cos\iota}{\cos\iota'}$$

$$\therefore \tan\alpha_2 = \frac{e\sin(\alpha-\iota)\cos\iota'}{\cos(\alpha-\iota)\cos\iota' - 2(1+e)\sin(\alpha-\iota)\sin\iota'}$$

$$= \frac{e\tan(\alpha-\iota)}{1 - 2(1+e)\tan(\alpha-\iota)\tan\iota'} = \frac{(1-e)e\tan(\alpha-\iota)}{1-e-2(1-e^2)\tan(\alpha-\iota)\tan\iota}$$

$$\text{wherefore } \tan\beta_2 = \frac{(1-e)e\tan(\alpha-\iota)}{1-e-2(1-e)\tan(\alpha-\iota)\tan\iota}$$

$$\tan\beta_3 = \frac{(1-e)e^2\tan(\alpha-\iota)}{1-e-2(1-e^2)\tan(\alpha-\iota)\tan\iota}$$

$$\text{similarly } \tan\beta_4 = \frac{(1-e)e^3\tan(\alpha-\iota)}{1-e-2(1-e^3)\tan(\alpha-\iota)\tan\iota}$$

$$\tan\beta_x = \frac{(1-e)e^{x-1}\tan(\alpha-\iota)}{1-e-2(1-e^{x-1})\tan(\alpha-\iota)\tan\iota}$$

Retaining the figure of last problem we have.

$t_1 = \frac{2v \sin\alpha'}{g\cos i}$, and the constant horizontal velocity $= v\cos(\alpha+i)$

$\therefore$ $d.r = \frac{2v^2 \sin\alpha \cos(\alpha+i)}{g\cos i}$

$\therefore$ $d.a = \frac{2v^2 d\alpha \cos(\alpha+i)}{g\cos^2 i} = R_1$

Similarly $t_2 = \frac{2ue \sin(\alpha+\beta)}{g\cos i}$ $\therefore$ $d.b_1 = \frac{2v'e d\alpha \times v'}{g\cos^2 i} = R_2$

for the horizontal velocity during the second range $=$

$-ue \sin\alpha \sin i' + \{v\cos\alpha - 2v \frac{\sin\alpha d\alpha}{\cos i'}\}\cos i$

$= v\{(\cos\alpha\cos i - 2d\alpha \sin i') - e \sin\alpha \sin i\}. = v'$

$\therefore$ $v d' = \frac{2v^2 e \sin\alpha\{\cos\alpha\cos i - (2+e)\sin\alpha\sin i\}}{g\cos^2 i}$.

$= \frac{2v^2 e d\alpha}{g\cos^2 i}\{\cos(\alpha+i) - (1+e)\sin\alpha\sin i\} = R_2$

Again $t_3 = \frac{2ve^2 \sin\alpha}{g\cos i}$

and horizontal velocity during the 3rd range $=$

$-ue^2 \sin\alpha\sin i + v\cos\alpha\cos i - 2v\sin\alpha d\alpha - 2ue\sin\alpha\cos i$

$= v\{\cos(\alpha+i) - \{1+2e+e^2\}(d\alpha\sin\alpha\sin i')\}$

$\therefore$ $v'' = \frac{2v^2 e^2 \sin\alpha}{g\cos^2 i}\{\cos(\alpha+i) - (1+e)^2\sin\alpha d.i\}. = R_3$.

Similarly

$$R_2 = \frac{2v^2 e^{n-1} \sin\alpha}{g \cos^2 i} \left\{ \cos(\alpha+i) - \left\{ 1 + 2e + 2e^2 + \cdots + e^{n-1} \right\} \sin\alpha \sin i \right\}$$

$$R_{n+1} = \frac{2v^2 e^n \sin\alpha}{g \cos^2 i} \left\{ \cos(\alpha+i) - \left\{ 1 + 2e + 2e^2 + \cdots + e^n \right\} \sin\alpha \sin i \right\}.$$

$$R_{n+2} = \frac{2v^2 e^{n+1} \sin\alpha}{g \cos^2 i} \left\{ \cos(\alpha+i) - \left(1 + 2e + 2e^2 + \cdots + e^{n+1} \right) \sin\alpha \sin i \right\}$$

$$\therefore \quad R_{n+2} - e(1+e) R_{n+1} + e^2 R_n =$$

$$\frac{2v^2 \sin\alpha \cdot e^{n-1}}{g \cos^2 i} \left\{ \begin{array}{l} e^2 \cos(\alpha+i) - \left\{ e^2 + 2e^3 + \cdots + e^{n+3} \right\} \sin\alpha \sin i \\ - e^2(1+e) \cos(\alpha+i) + (1+e) \left\{ e^2 + 2e^3 + \cdots + e^{n+3} \right\} \sin\alpha \sin i \\ + e^3 \cos(\alpha+i) - e \left\{ e^2 + 2e^3 + \cdots + e^{n+3} \right\} \sin\alpha \sin i \end{array} \right\}$$

$$= \frac{2v^2 \sin\alpha \, e^{n-1}}{g \cos^2 i} \times 0 = 0$$

$r = a\left(\sec\frac{\theta}{n}\right)^n \quad \therefore u = \frac{1}{a}\left(\cos\frac{\theta}{n}\right)^n.$

$\log u = n \log\cos\left(\frac{\theta}{n}\right). \qquad \left(\frac{r}{a}\right)^{\frac{2}{n}} = \sec\frac{\theta}{n}$

$$\frac{du}{u} = -\frac{n}{n}\cdot\frac{\sin\frac{\theta}{n}}{\cos\frac{\theta}{n}} = -\tan\frac{\theta}{n}.$$

$$\therefore \frac{du}{d\theta} = -u\tan\left(\frac{\theta}{n}\right).$$

$$\frac{d^2u}{d\theta^2} = -\frac{du}{d\theta}\tan\frac{\theta}{n} - u\frac{\sec^2\frac{\theta}{n}}{n}$$

$$= u\tan^2\frac{\theta}{n} - \frac{u}{n}\sec^2\frac{\theta}{n}.$$

$$= u\sec^2\left(\frac{\theta}{n}\right) - u - \frac{u}{n}\sec^2\frac{\theta}{n}.$$

$$\therefore \frac{d^2u}{d\theta^2} = u\left(1-\frac{1}{n}\right)\sec^2\frac{\theta}{n}.$$

$$F = h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) = h^2u^3\left(\frac{n-1}{n}\right)\sec^2\frac{\theta}{n}.$$

$$= \frac{h^2}{r^3}\left(\frac{n-1}{n}\right)\sec^2\frac{\theta}{n}.$$

$$F = \frac{h^2}{r^3}\left(\frac{n-1}{n}\right)\frac{1}{a^{\frac{2}{n}}}\cdot r^{\frac{2}{n}}.$$

$$= \frac{h^2}{r^{\frac{3n-2}{3n}}}\ \frac{n-1}{n}\ \frac{1}{a^{\frac{2}{n}}}.$$

$$F = \frac{h^2}{a^{\frac{2}{n}}}\ \frac{n-1}{n}\cdot\frac{1}{r^{\frac{3n-2}{3n}}}.$$

$r = a\left(\sec\frac{\theta}{n}\right)^n \quad \therefore u = \frac{1}{a}\left(\cos\frac{\theta}{n}\right)^n.$

$\log u = n \log\cos\left(\frac{\theta}{n}\right). \qquad \left(\frac{r}{a}\right)^{\frac{2}{n}} = \sec\frac{\theta}{n}$

$$\frac{du}{u} = -\frac{n}{n}\cdot\frac{\sin\frac{\theta}{n}}{\cos\frac{\theta}{n}} = -\tan\frac{\theta}{n}.$$

$$\therefore \frac{du}{d\theta} = -u\tan\left(\frac{\theta}{n}\right).$$

$$\frac{d^2u}{d\theta^2} = -\frac{du}{d\theta}\tan\frac{\theta}{n} - u\frac{\sec^2\frac{\theta}{n}}{n}$$

$$= u\tan^2\frac{\theta}{n} - \frac{u}{n}\sec^2\frac{\theta}{n}.$$

$$= u\sec^2\left(\frac{\theta}{n}\right) - u - \frac{u}{n}\sec^2\frac{\theta}{n}.$$

$$\therefore \frac{d^2u}{d\theta^2} = u\left(1-\frac{1}{n}\right)\sec^2\frac{\theta}{n}.$$

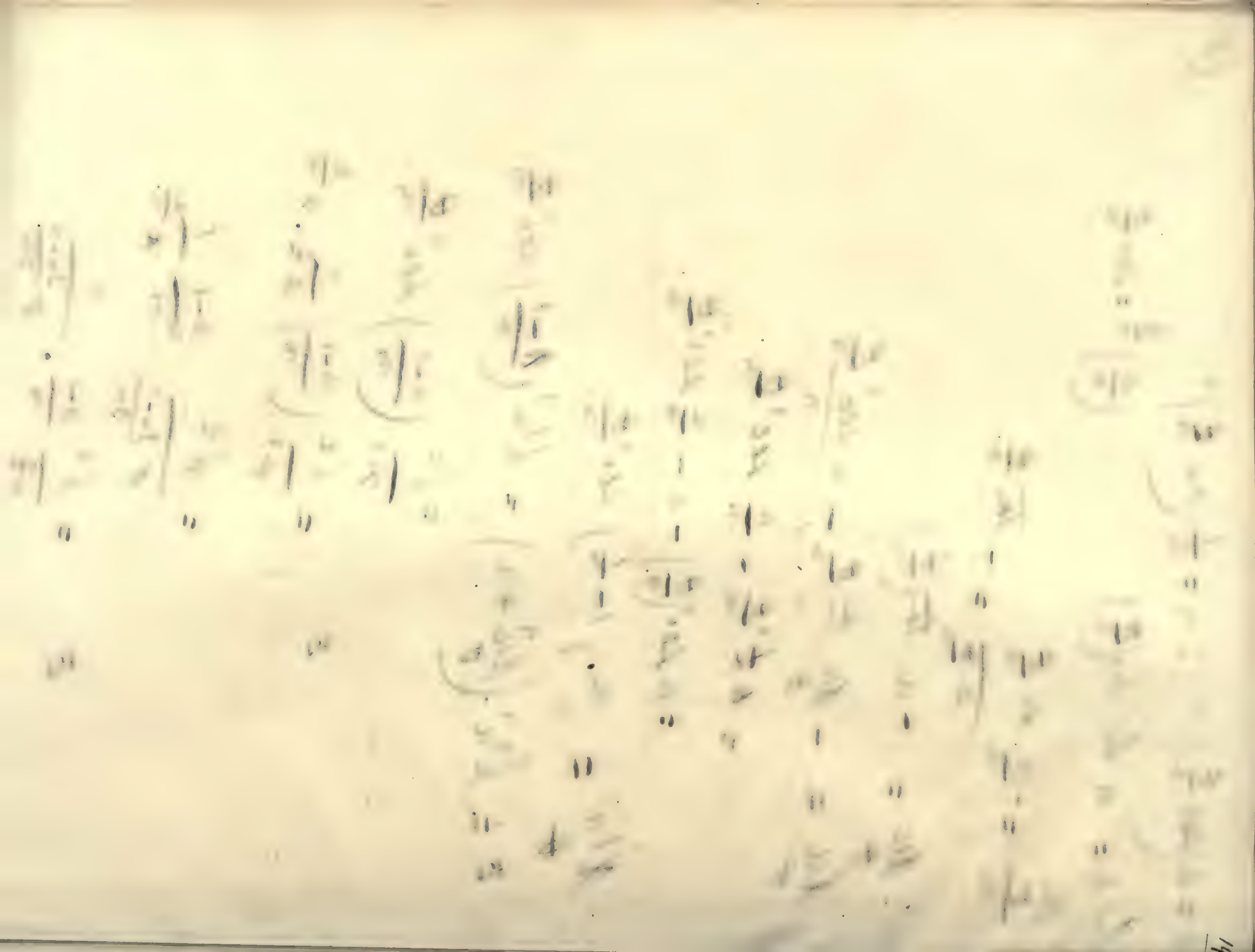
$$E = h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) = h^2u^3\left(\frac{n-1}{n}\right)\sec^2\frac{\theta}{n}.$$

$$= \frac{h^2}{r^3}\left(\frac{n-1}{n}\right)\sec^2\frac{\theta}{n}.$$

$$E = \frac{h^2}{r^3}\left(\frac{n-1}{n}\right)\frac{1}{a^{\frac{2}{n}}}\cdot r^{\frac{2}{n}}.$$

$$= \frac{h^2}{r^{\frac{3n-2}{3n}}}\,\frac{n-1}{n}\,\frac{1}{a^{\frac{2}{n}}}.$$

$$E = \frac{h^2}{a^{\frac{2}{n}}}\,\frac{n-1}{n}\cdot\frac{1}{r^{\frac{3n-2}{3n}}}.$$



$$\frac{2u\cdot(1+\cot^2\theta)}{\cot\theta\cdot\operatorname{cosec}^2\theta} = \frac{\cot\theta}{a\sin^4\theta}$$

$$u = \frac{1}{2a}\cot^2\theta\operatorname{cosec}\theta.$$

$$\frac{du}{d\theta} = \frac{1}{2a}\left(-\cot^2\theta\cdot\operatorname{cosec}\theta\cot\theta + \operatorname{cosec}^3\theta\right)$$

$$1 + \frac{1}{} = \left(-\operatorname{cosec}^2\theta - \cot^2\theta\right)\operatorname{cosec}\theta\cdot\cot\theta$$

$$= \frac{1}{2a}\cdot\operatorname{cosec}\theta = \left(1+2\cot^2\theta\right)\operatorname{cosec}\theta$$

$$\frac{d^2u}{d\theta^2} = \left(1+2\cot^2\theta\right)\cot\theta\operatorname{cosec}\theta + \operatorname{cosec}^3\theta\left(4\cot\theta\right.$$

$$= \left(2\operatorname{cosec}^2\theta - 1\right)\cot\theta\operatorname{cosec}\theta + 4\operatorname{cosec}^2\theta\cot\theta.$$

$$u = = \left(6\operatorname{cosec}^2\theta - 1\right)\cot\theta\operatorname{cosec}\theta$$

$$\frac{d^2u}{d\theta^2} + u = 6\operatorname{cosec}^2\theta\cdot\left(\cot\theta\operatorname{cosec}\theta\right)$$

$$= \frac{3}{a}\operatorname{cosec}^2\theta\cdot u$$

$$E = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = \frac{3h^2}{a}\cdot u^3\operatorname{cosec}^2\theta$$

$$= \frac{3h^2}{a}\cdot\frac{\operatorname{cosec}^2\theta}{r^3}$$

$$y = \frac{r^{\frac{3}{2}}}{(2a-r)^{\frac{1}{2}}}$$

$$r^2 \sin^2\theta = \frac{r^3 \cos^3\theta}{2a - r\cos\theta}.$$

$$(2a - r\cos\theta)\sin^2\theta = r\cos^3\theta$$

$$= 2a\sin^2\theta \cdot r(\cos^3\theta + \cos\theta\sin^2\theta)$$

$$\therefore r = \frac{2a\sin^2\theta}{\cos\theta}.$$

$$u = \frac{\cos\theta}{2a\sin^2\theta}$$

$$\frac{du}{d\theta} = \frac{-2a\sin^3\theta - \cos\theta \cdot 4a\sin\theta\cos\theta - 2a\sin\theta(\sin^2\theta + \cos^2\theta)}{4a^2\sin^4\theta} - 2a\cos^2\theta.$$

$$= -2a\left(1 + \cos^2\theta\right) - \frac{2a(1+\cos^2\theta)}{4a^2\sin^3\theta}$$

$$\frac{\cos\theta}{2a\sin^2\theta}$$

$$\therefore \frac{du}{d\theta} = -\frac{u(1+\cos^2\theta)}{\cos\theta\sin\theta} = \frac{4a^2\sin\theta(1+\cos^2\theta)}{4a^2\sin^3\theta\cos\theta}$$

$$= -\frac{1+\cos^2\theta}{\cos\theta}$$

$$u = \frac{1}{2a} \cdot \cot\theta \cdot \csc\theta.$$

$$\log u = \frac{1}{2a}$$

$$=$$

$r = 2a \cos\theta$.

$u = \frac{1}{2a} \sec\theta$.

$\log u = \log \sec\theta - \log 2a$.

$\frac{du}{d\theta} \cdot \frac{1}{u} = \frac{\tan\theta \sec\theta}{\sec\theta} = \tan\theta$

$\frac{du}{d\theta} = u \tan\theta$.

$\frac{d^2u}{d\theta^2} = u \tan^2\theta + u \sec^2\theta$

$= u(2\sec^2\theta - 1)$

$\therefore u + \frac{d^2u}{d\theta^2} = 2u \sec^2\theta = 8a^2u^3$

$\therefore F = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$

$= h^2u^2(8a^2u^3) = 8a^2h^2u^5$

$= \frac{8a^2h^2}{r^5} = \frac{\mu}{r^5}$

also $\frac{F.r}{r} = V^2$

$\therefore \frac{4a^2h^2}{r^4} = V^2$

$\therefore V = \frac{2ah}{r^2}$

$V^2 = \frac{4a^2h^2}{r^4} = \frac{8a^2h^2}{2r^4} = \frac{\mu}{2r^4}$.

(3)

$$h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) = \frac{2a^2n}{r^5} + \frac{n}{r^3}.$$

Now the velocity in a circle at distance $a = \sqrt{\frac{3n}{a}}$

$\therefore$ velocity of projection $= \frac{\sqrt{2n}}{a}$.

also $h^2 = a^2(\text{velocity of proj})^2 \sin^2 45° \cdot \frac{2n}{a} = n$.

$$\therefore \quad nu^2\left(\frac{d^2u}{d\theta^2}+u\right) = 2a^2u^5n + nu^3.$$

$$\frac{d^2u}{d\theta^2} = 2a^2u^3$$

$$\left(\frac{du}{d\theta}\right)^2 = a^2u^4 + C.$$

Now $\frac{u}{\frac{du}{d\theta}} = \tan SPL = 1 \quad \therefore \left(\frac{du}{d\theta}\right)^2 = u^2 = \frac{1}{a^2}$ at the point

$$\therefore \frac{1}{a^4} = \frac{a^2}{a^4} + C \quad \therefore C = 0.$$

$$\frac{du}{d\theta} = au^2.$$

$$\frac{d\theta}{du} = \frac{1}{au^2} \quad \therefore \theta = -\frac{1}{au} + C.$$

when $\theta = 0$ $u = \frac{1}{a}$ $\therefore 0 = -1 + C$ or $C = 1$

$$\theta = -\frac{r}{a} + 1$$

$$\therefore r = a(1-\theta)$$

μ = absolute $\odot$ $\therefore F = \frac{\mu}{r^2}$

But $F = -\frac{h^2}{2} d_r \frac{1}{r^2} = \frac{h^2}{r} \frac{1}{r^2}$ and therefore $\mu = \frac{h^2}{a}$ after the $\odot$ has been increased when

$$F = \frac{mh^2}{a} \cdot \frac{1}{r^2}$$

$$\therefore \frac{mh^2}{a} \frac{1}{r^2} = \frac{h^2}{\mu^3} d_r \mu \quad \text{or} \quad \frac{m}{a} \frac{1}{r^2} = \frac{1}{\mu^3} d_r \mu$$

Integrate and when $\frac{m}{a} \frac{1}{r} = C + \frac{1}{2\mu^2}$.

when the absolute $\odot$ is increased $r = a$, also μ is common to both orbits

$$\therefore \frac{m}{a^2} = C + \frac{1}{2a^2} \quad \therefore C = \frac{2m-1}{2a^2}$$

$$\text{and } \frac{m}{a} \cdot \frac{1}{r} = \frac{2m-1}{2a^2} + \frac{1}{2\mu^2}$$

$$\frac{a^2}{2m-1} \frac{1}{\mu^2} = \frac{2ma}{2m-1} \frac{1}{r} - 1$$

If we compare this with $= r$ of the $\odot$

$$\frac{b^2}{\mu^2} = \frac{2a'}{r} - 1 \quad \text{we have}$$

$$b' = \frac{a}{\sqrt{2m-1}}, \quad a' = \frac{ma}{2m-1}$$

$$\varepsilon = \frac{\sqrt{a'^2 - b'^2}}{a'} = \frac{\sqrt{m^2a^2 - (2m-1)a^2}}{ma} = \frac{\sqrt{m^2a^2 - 2ma^2 + a^2}}{ma}$$

$$= \frac{m-1}{m}$$

$$r = a(1+\varepsilon t)$$

$$d_f r = r(d+\theta)\hat{} - \not{\text{R}}$$

$$r\hat{}\, d+\theta = h$$

$$\therefore \quad d_f r = \frac{h\hat{}}{r\hat{}} - \frac{r-a}{a\varepsilon}$$

$$(d+w)\hat{} = c - \frac{h\hat{}}{r\hat{}} - \frac{(r-a)\hat{}}{a\varepsilon}$$

$$0 = c - \frac{h\hat{}}{a\hat{}}$$

$$0 = \frac{h\hat{}}{a\hat{}} - \frac{h\hat{}}{r\hat{}} - \frac{(r-a)\hat{}}{a\varepsilon}$$

$$\frac{h\hat{}}{a\hat{}\, r\hat{}}(r\hat{}-a\hat{}) = \frac{(r-a)\hat{}}{a\varepsilon}$$

$$\frac{h\hat{}}{a\, r^2} = \varepsilon' \cdot \frac{r-a}{r+a}$$

$$h = a^2\omega \qquad \frac{a^3\cdot\omega^2\varepsilon}{r^2} = \frac{r-a}{r+a}$$

$$4a\hat{}\omega\hat{} = (2a)\cdot T \qquad T\cdot\varepsilon = 1$$

$$2a = a(1+\varepsilon T)$$

$$2a\omega\hat{} = T = \varepsilon'$$

$$\omega\hat{}\varepsilon = \frac{1}{2a}$$

$$\frac{a\hat{}}{r^2} = 2\cdot\frac{r-a}{r+a}$$

we have for the motion at any point

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2} \quad \therefore \left(\frac{dx}{dt}\right)^2 = 2\mu \cdot \frac{1}{x} + C.$$

for the internal space $\frac{dx}{dt} = 0$ when $x = r$,

and for the external space $\frac{dx}{dt} = 0$ when $x' = \infty$

$\therefore$ if v^2 be the velocity at the point of the curve

we have $v^2 = 2\mu\left(\frac{1}{x} - \frac{1}{r}\right) = 2\mu\cdot\left(\frac{1}{r} - \frac{1}{x}\right)$.

But the $\left(\frac{\text{velocity}}{2E}\right)^2 = \frac{\mu}{4}$ where μ = Chord of curve.

$$\therefore v^2 = \frac{\mu}{2}\cdot\frac{\mu}{r^2}$$

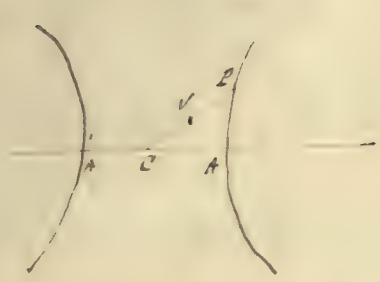
$$\therefore \frac{\mu}{4r^2} = \frac{1}{x} - \frac{1}{r} \quad \therefore \frac{1}{x} = \frac{4r+\mu}{4r^2} \text{ or } x = \frac{4r^2}{4r+\mu}.$$

$$\text{and } r - \frac{4r^2}{4r+\mu} = \frac{\mu r}{4r+\mu} = c'$$

similarly we have $\frac{\mu r}{4r-\mu} = c$.

$$\therefore \frac{c-c'}{c+c'} = \frac{\frac{1}{4r-\mu} - \frac{1}{4r+\mu}}{\frac{1}{4r-\mu} + \frac{1}{4r+\mu}} = \frac{2\mu}{8r} = \frac{\mu}{4r}$$

$$\therefore \mu = 4r\cdot\frac{c-c'}{c+c'}$$



L the portn of the surface abt V

$r = \frac{a}{\text{vers } 2\alpha}$ the = n bute ft.

Now the space dr is inverty as L

equal $\frac{1}{2}$ Chord of curv. at L thro C = $\frac{1}{2}$ CL (for CV = VL).

∴ y CL the half = $\frac{1}{2}$ CL in wch be a point in the

required locus and CI = $r' = \frac{1}{2} \frac{a}{\text{vers } \alpha}$ in the = n+1

which is one point in the locus

∴ the required locus is an epicycloid ft.

having its diameter = $\frac{1}{2}a$, a-d its axis coincident

with that of the original ft.

~~Eqn to prob path~~

$f a = a \cdot \varphi\left(\frac{1}{a}\right)$

$d_\beta u + a = \frac{1}{h} \cdot \varphi\left(\frac{1}{\gamma}\right)$

Let $r = a - x$ when x is small

$\therefore \varphi\left(\frac{1}{\gamma}\right) = \varphi\left(\frac{1}{a}\right) + \frac{x}{a^2} \cdot \varphi'\left(\frac{1}{a}\right)$ neglecting higher powers

$\varphi\left(\frac{1}{a-x}\right) = \varphi\left(\frac{1}{a} + \frac{x}{a^2}\right)$

$= \varphi\left(\frac{1}{a}\right) + \frac{x}{a^2} \cdot \varphi'\left(\frac{1}{a}\right)$

$\therefore f u = \frac{1}{a-x} = \frac{1}{a} + \frac{x}{a^2}$

$d_\beta u = \frac{d_\beta x}{a^2}$

$\therefore \frac{d_\beta x}{a^2} + \frac{1}{a} + \frac{x}{a^2} = \frac{1}{h} \cdot \left\{ \varphi\left(\frac{1}{a}\right) + \frac{x}{a^2} \cdot \varphi'\left(\frac{1}{a}\right) \right\}$

$\frac{v^2}{(1+m)^2} = \frac{\frac{1}{a} \cdot \varphi\left(\frac{1}{a}\right)}{(1+m)^2}$

$\therefore h^2 = f \cdot r = \frac{a \cdot \varphi\left(\frac{1}{a}\right)}{(1+m)^2}$

$\therefore d_\beta x + a + r = \frac{a (1+m)}{a \cdot \varphi\left(\frac{1}{a}\right)} \left\{ \varphi\left(\frac{1}{a}\right) + \frac{x}{a} \cdot \varphi'\left(\frac{1}{a}\right) \right\}$

$= a \cdot (1+m)^2 + \frac{x}{a} \cdot 1+m \cdot \frac{\varphi'}{\varphi}$

$d_\beta x + x \left\{ 1 - \frac{1+m}{a} \cdot \frac{\varphi'}{\varphi} \right\} = a (1+m^2 - 1)$

$(d_\beta x) + x \left(1 - \frac{1+m}{a} \cdot \frac{\varphi'}{\varphi} \right) = 2a \cdot (1+m^2 - 1) x + \text{¢}$

$x = \frac{2a (1+m^2 - 1) \left\{ 1 + \frac{1+m}{a} \frac{\varphi'}{\varphi} \right\}}{1 - \frac{1+m}{a} \cdot \frac{\varphi'}{\varphi}}$

$= \frac{4 a m \varphi\left(\frac{1}{a}\right)}{\varphi\left(\frac{1}{a}\right) - \frac{1}{a} \cdot \varphi'\left(\frac{1}{a}\right)}$ $\therefore a' = a - x$

$= a \left\{ 1 - \frac{4 m \varphi\left(\frac{1}{a}\right)}{\varphi\left(\frac{1}{a}\right) - \frac{1}{a} \varphi'\left(\frac{1}{a}\right)} \right\}$

Take C for origin & CL for positive
y & the particle at time t. CM = x. ML = y
& Co-ord. of L. Then acc. equations

$$\ddot{x} = -\lambda x - g\frac{dy}{ds} \quad (1) \qquad \frac{v^2}{\rho} = +g\frac{dx}{ds} \quad (2)$$

from (2) $v^2 = +g\int dx = -g\left(\frac{dx}{dy}\right)^{-1}$. Now $y = \sqrt{a^2-x^2}$

$$= -g\left\{\frac{a^2}{\frac{-a^2}{(a^2-x^2)^{\frac{3}{2}}}}\right\} = g\sqrt{a^2-x^2} \quad (3)$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{-\sqrt{a^2-x^2}+x^2\cdot\frac{1}{2}\cdot\frac{-2x}{\sqrt{a^2-x^2}}}{a^2-x^2} = \frac{-a^2}{(a^2-x^2)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = \frac{3x}{(a^2-x^2)^{\frac{5}{2}}}$$

Now $v\frac{dv}{dx} = \frac{1}{2}g\cdot\frac{-x}{\sqrt{a^2-x^2}}$

$$\therefore \int v\,dv = -\frac{g}{2}\int\frac{x\,dx}{\sqrt{a^2-x^2}}$$

$$= -\frac{g}{2}\cdot\frac{2\sqrt{a^2-x^2}\cdot a}{a^2\sqrt{a^2-x^2}} = -\frac{gx}{2a}$$

$$\therefore \frac{gx}{2a}\sqrt{a^2-x^2} = R + g\int dy \quad \text{from (1)}$$

$$\frac{g^2}{R} = R + g\int dy\cdot\frac{dx}{dy}$$

$$= R + g\cdot\frac{-x}{\sqrt{a^2-x^2}}\cdot\frac{\sqrt{a^2-x^2}}{a} = R - g\frac{x}{a}.$$

$$\therefore R = \frac{3}{2}\cdot\frac{gx}{a} \qquad \text{---------- (4).}$$

$$\int dx\,dR = \int g\,dx$$

$$\therefore \frac{3}{2}\cdot\frac{gx}{a} = 2g\sqrt{a^2-x^2}.$$

$$\therefore x = \frac{3x}{2a}\cdot(a^2-x^2)^{-\frac{1}{2}} \qquad \text{---------- (5)}$$

$$\therefore R = \frac{3}{2}\cdot\frac{gx}{a}, \qquad v^2 = g\sqrt{a^2-x^2}, \qquad x = \frac{3x}{2a}\cdot(a^2-x^2)^{-\frac{1}{2}}.$$

Letter = [illegible] $y = b - cx^n$.

Ll = [illegible] ac. $v\,dv = -R\,dy$, $v^2 = \int R\,dx$ (2.

$$\left.\begin{aligned} y &= b - cx^n \\ \frac{dy}{dx} &= -ncx^{n-1} \\ \frac{ds}{dx} &= \sqrt{1+n^2c^2x^{2n-2}} \\ \frac{d^2y}{dx^2} &= -n(n-1)c\,x^{n-2} \end{aligned}\right\}$$

$$\text{Now } v^2 = -g\cdot\frac{\left(\frac{dx}{dy}\right)^2}{\frac{d^2y}{dx^2}} = g\left\{\frac{1+n^2c^2x^{2n-2}}{n(n-1)cx^{n-2}}\right\}\cdot x^{n-2}$$

$$= \frac{g}{n(n-1)c}\cdot\frac{1+n^2c^2x^{n-2}}{x^{n-2}}.$$

$$\therefore v\frac{dv}{dx} = \frac{g}{2n(n-1)c}\left\{\frac{x^{n-1}\left\{n^2c^2\cdot(2n-2)x^{2n-3}\right\} - \left(1+n^2c^2x^{2n-2}\right)(n-2)x^{n-3}}{x^{2n-4}}\right.$$

$$= \frac{g}{2n(n-1)c}\cdot\left\{\frac{(2n-2)n^2c^2x^{3n-4} - (n-2)x^{n-3} - (n-2)n^2c^2x^{3n-4}}{x^{2n-4}}\right.$$

$$= \frac{g}{2n(n-1)c}\cdot\frac{n^2c^2x^{2n-2} - (n-2)}{x^{n-1}}.$$

$$\therefore v\,dv = \frac{g}{2n(n-1)c}\left\{\frac{n^2c^2x^{2n-2} - (n-2)}{x^{n-1}\sqrt{1+n^2c^2x^{2n-2}}}\right\}.$$

$$\therefore R = -\left\{v\frac{dv}{dy} + \frac{g}{2n(n-1)c}\cdot\frac{n^2c^2x^{2n-2} - (n-2)}{x^{n-1}\sqrt{1+n^2c^2x^{2n-2}}}\right.$$

$$= -g\left\{\frac{-ncx^{n-1}}{\sqrt{1+n^2c^2x^{2n-2}}} + \frac{n^2c^2x^{2n-2} - (n-2)}{2n(n-1)cx^{n-1}\sqrt{1+n^2c^2x^{2n-2}}}\right\}$$

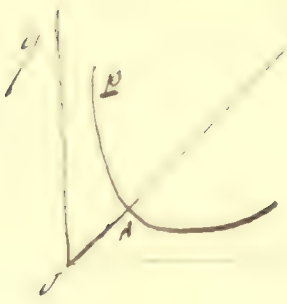
$$= -g\left\{\frac{-2n^2c^2 + 2n^2c^2}{2n(n-1)c^2x^{n-1}\sqrt{1+n^2c^2x^{2n-2}}}x^{2n-2} + n^2c^2x^{2n-2} - (n-2)\right\}$$

$$= -g\left\{\frac{-n^2c^2x^{2n-2} + n^2c^2x^{2n-2} - (n-2)}{2n(n-1)c^2x^{n-1}\sqrt{1+n^2c^2x^{2n-2}}}\right\}.$$

$$= \frac{g}{2n(n-1)c}\left\{\frac{(n-2)\left(n^2c^2x^{2n-2}+1\right)}{x^{n-1}\sqrt{1+n^2c^2x^{2n-2}}}\right\} = \frac{g(n-2)\sqrt{1+n^2c^2x^{2n-2}}}{2n(n-1)c\cdot x^{n-1}}$$

$\therefore$ $M\frac{dv}{dt} = M\frac{v\,dv}{ds} = -R - Mg\frac{dy}{ds}$ (1)

$v^2 = \frac{g}{l}\cdot\frac{ds^2}{dt^2}$ (2)



From the fact

$$R = -\frac{g}{2}\cdot\frac{ds}{dx}\cdot d_n\left(\frac{1}{d_n^2 y}\right)$$

$$\therefore R = -\frac{g}{2}\cdot\frac{\sqrt{x^{2n+2} + \{ax^{n+1} - ny^{n+1}\}^2(n+2)}}{n(n+1)y^{n+1}}.$$

$$v^2 = -g\cdot\frac{\left(\frac{ds}{dx}\right)^2}{d_n^2 y}$$

$$= -g\left\{\frac{x^{2n+2} + \{ax^{n+1} - ny^{n+1}\}^2\, x^{n+2}}{x^{2n+2}\,n(n+1)y^{n+1}}\right\}$$

$$= -g\left\{\frac{x^{2n+2} + \{ax^{n+1} - ny^{n+1}\}^2}{n(n+1)y^{n+1}x^n}\right\}.$$

But $av^2 = R$ $\therefore$ $a = \frac{R}{v^2}$

$$a = \frac{1}{2}\left\{\frac{n+2\,(x)^n}{x^{2n+2} + (ax^{n+1} - ny^{n+1})^2}\right\}^{\frac{1}{2}}.$$

$y = ax + \frac{x^{n+1}}{x^n}$

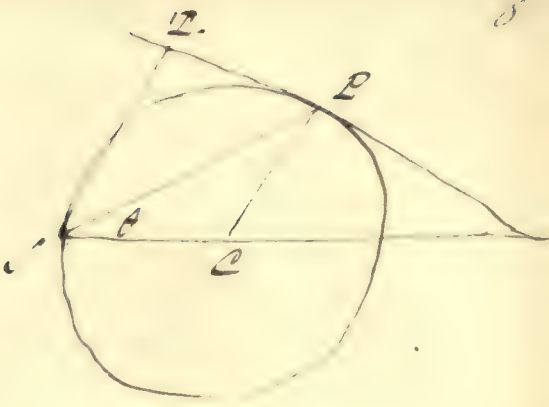
$d_x y = a - \frac{n\cdot x^{n+1}}{x^{n+1}}$

$d_x^2 y = +\frac{n(n+1)x^{n+1}}{x^{n+2}}$

$\therefore \frac{1}{d_n^2 y} = \frac{x^{n+2}}{n(n+1)x^{n+1}}$

$d_n\left(\frac{1}{d_n^2 y}\right) = \frac{(n+2)x}{n(n+1)x^{n+1}}$

$d_x s = \sqrt{1 + \left(\frac{ax^{n+1} - ny^{n+1}}{x^{n+1}}\right)^2}\,dx$



Ult cent. force. $= P = r \cdot \sqrt{\Pi} = p$.

$\therefore \frac{1}{2} =$ [illegible] motion acc.

$$\therefore dv = -f - \mu r^n dr$$

$$v^2 = \int \mu r^n \frac{dr}{ds} \frac{h}{p} = \mu r^n \int d_p r.$$

$$\therefore 2v \frac{dv}{ds} = d_s \left\{ \frac{1}{p^2} \cdot p^3 \frac{dr}{dp} \cdot \mu r^n \right\} = -2\mu r^n \frac{dr}{ds} + \frac{1}{p^2} d_s \left\{ p^3 \frac{dr}{dp} \mu r^{n-1} \right\}.$$

$$\therefore \quad f = -\frac{1}{2p^2} \cdot d_s \left\{ p^3 \frac{dr}{d\theta} \cdot \mu r^{n-1} \right\} = -\frac{1}{2p^2} \cdot d_\theta \left\{ p^3 \frac{dr}{d\theta} \cdot \mu r^{n-1} \right\} \frac{d\theta}{ds}$$

$$\frac{d\theta}{ds} = \frac{d\theta}{\frac{dx}{\cos\theta}} = -\frac{1}{2a}$$

$$\text{Also } \frac{dr}{d\theta} = \frac{-2a \sin\theta}{-4a \cos\theta \sin\theta} = \frac{1}{2\cos\theta}$$

for $x = 2a\cos^2\theta \quad \therefore dx = -4a\cos\theta \sin\theta \, d\theta$

$$\frac{ds}{dx} = \frac{a}{y} = \frac{a}{2a \sin\theta \cos\theta}$$

$$\therefore \frac{d\theta}{ds} = \frac{2\sin\theta\cos\theta}{4a\cos\theta\sin\theta} = -\frac{1}{2a}.$$

$$\therefore f = -\frac{1}{2p^2} \cdot d_\theta \left\{ p^3 \frac{dr}{d\theta} \cdot \mu r^n \right\} \frac{d\theta}{ds}$$

$$= -\frac{1}{2 \cdot (2a\cos^2\theta)^2} d_\theta \left\{ (2a)^3 \cos^6\theta \cdot \frac{1}{2\cos\theta} \cdot \mu \cdot (2a)^n \cos^n\theta \right\} \times -\frac{1}{2a}.$$

$$= \frac{1}{4} \cdot \frac{1}{(2a)^3 \cos^2\theta} \cdot \left\{ (2a)^{n+3} \cdot (n+5) \mu \cos^{n+4}\theta \right\}.$$

$$= \frac{\mu(n+5)}{4} \cdot (2a)^n \cos^n\theta = \frac{\mu}{4} \cdot (n+5) r^n.$$

$$\text{Again } \frac{v^2}{\rho} = \mu r^n \cdot \frac{1}{p} \quad \text{or } v^2 = \mu r^n \cdot \frac{r}{p^2}.$$

$$\frac{dp}{p} = \frac{1}{r^2 \cos\theta} \quad \therefore \int \frac{dp}{p} = \int \sec\theta = \frac{1}{2} r$$

$$\therefore v^2 = \frac{\mu}{2} r^{n+1}$$

$$\text{But } dv^2 = R.$$

$$\therefore a = \frac{R}{v^2} = \frac{\frac{\mu}{4} \cdot (n+1) r^n}{\frac{\mu}{2} r^{n+1}}.$$

$$= \frac{\mu}{2} \cdot (n+1) \cdot \frac{1}{r}$$

Δu = u [illegible] Δt = $r \Delta x$ [illegible]

$$\Delta^2 = \frac{1}{2} \frac{1}{\Delta t^2} \cdot \Delta_2 \left(\mu^0 \Delta x^n \right) = \frac{1}{2} \cdot \Delta_2 \left(\mu^3 \frac{1}{\Delta x^3} \cdot \Delta x^{n} \right)$$

$$= \frac{1}{2^2} \Delta^2 \left(\mu x^{n+3} \cdot \Delta^2 x \right)$$

$$= \frac{\mu}{2^2 \Delta x^2} \cdot \Delta x \left(x^{n+3} \right) \cdot \Delta x$$

$$= \frac{\mu}{2} (n+3) x^n \cdot \Delta x$$

also $v^2 = \Delta x^n \cdot \frac{1}{2} \Delta x = \Delta x^{n+1}$

$$\therefore \alpha^2 = \frac{\rho}{\gamma a} = \frac{\mu}{2} \cdot (n+3) \frac{\Delta x}{x}$$

We have for the motion

$$v\,dv = -\mu - P\,dr \qquad (1)$$

$$\frac{v^2}{\rho} = P\frac{r}{\rho} \qquad (2)$$

From (1) $\frac{1}{2}v^2 = -\mu s - Pr + C$.

At the commencement of motion $s = 0$, $v = \beta$.

$\therefore \frac{1}{2}\beta^2 = -Pr + C$.

$\therefore \frac{1}{2}v^2 = \frac{1}{2}\beta^2 - \mu s$ or $v^2 = \beta^2 - 2\mu s$. — (3).

Also from (2). since $\mu = r$ we have.

$$P = \frac{v^2}{2} = \frac{1}{4}\{\beta^2 - 2\mu s\} \qquad (4)$$

Again when the particle is reduced to rest $v = 0$

$$\therefore \beta^2 - 2\mu s = 0 \text{ or } s = \frac{\beta^2}{2\mu} \qquad (5)$$

Lastly since the force varies inversely as the square of the velocity of projection, if t be the time that elapses before the particle is reduced to rest, since the resistance is μ we have $\mu t = \beta$

$$\therefore t = \frac{\beta}{\mu} \qquad (6)$$

$$\frac{d^2x}{dt^2} = -R.\cos\alpha = -\mu\frac{dx}{dt} = -\mu\frac{ds}{dt}\cdot\cos\alpha = -\mu\frac{dx}{dt}$$

$$\frac{d^2y}{dt^2} = -g - \mu\frac{dy}{dt}$$

$$\frac{dx}{dt} = C - \mu x \qquad (1)$$

$$\frac{dy}{dt} = C_1 - gt - \mu y \qquad (2)$$

$$u\cos\alpha = C$$

$$u\sin\alpha = C_1$$

$$\therefore \frac{dy}{dx} = \frac{u\sin\alpha - gt - \mu y}{u\cos\alpha - \mu x} = \tan\beta$$

$$\frac{dx}{dt} = u\cos\alpha - \mu x$$

$$\mu t = C - \log(u\cos\alpha - \mu x) = \log\left(\frac{C}{u\cos\alpha - \mu x}\right)$$

$$\mu t = \log\left(\frac{u\cos\alpha}{u\cos\alpha - \mu x}\right)$$

$$1 - \frac{\mu x}{u\cos\alpha} = \varepsilon^{-\mu t}$$

$$\mu x = (1 - \varepsilon^{-\mu t})\, u\cos\alpha$$

$$\frac{dy}{dt} + \mu y = u\sin\alpha - gt$$

$$\frac{d}{dt}(y\cdot\varepsilon^{\mu t}) = \varepsilon^{\mu t}(u\sin\alpha - gt)$$

$$y\cdot\varepsilon^{\mu t} = \frac{u\sin\alpha}{\mu}\cdot\varepsilon^{\mu t} - g\,\varepsilon^{\mu t}\cdot\left(\frac{t}{\mu} - \frac{1}{\mu^2}\right) + C$$

$$\mu y = u\sin\alpha - g\cdot\frac{\mu t - 1}{\mu} + C\cdot\varepsilon^{-\mu t}$$

$$0 = u\sin\alpha + \frac{g}{\mu} + C$$

$$\mu y = u\sin\alpha - gt + \frac{g}{\mu}(1 - \varepsilon^{-\mu t}) - u\sin\alpha\cdot\varepsilon^{-\mu t}$$

$$\tan\beta = \frac{u\sin\alpha \cdot \varepsilon^{-\mu t} - \frac{g}{\mu}(1-\varepsilon^{-\mu t})}{u\cos\alpha \cdot \varepsilon^{-\mu t}}$$

$$= \frac{u\sin\alpha - \frac{g}{\mu}(\varepsilon^{\mu t}-1)}{u\cos\alpha}$$

$$= \tan\alpha - \frac{g(\varepsilon^{\mu t}-1)}{\mu u\cos\alpha}$$

$$\frac{g\cdot(\varepsilon^{\mu t}-1)}{\mu u\cos\alpha} = \tan\alpha - \tan\beta$$

$$= \frac{\sin\alpha-\beta}{\cos\beta}$$

$$g\cos\beta(\varepsilon^{\mu t}-1) = \mu u\sin\alpha-\beta$$

$$\frac{d^2x}{dt^2} = -k\frac{dx}{ds} = -\mu \cdot \frac{dx}{dt} \cdot \frac{ds}{dt} = -\mu \frac{dx}{dt}.$$

$$\frac{dx}{dt} = -\mu x + c.$$

$u\cos\alpha = c \quad \therefore \frac{dx}{dt} = u\cos\alpha - \mu x.$

$$\frac{d^2y}{dt^2} = u - \mu \frac{dy}{dt}.$$

$$\frac{dy}{dt} = -gt - \mu y + c. \quad \frac{dy}{dt} = u\sin\alpha + gt - \mu y.$$

$$\therefore dy = \frac{u\sin\alpha + gt - \mu y}{u\cos\alpha - \mu x}.$$ [illegible]

And $\frac{dx}{dt} = u\cos\alpha - \mu x \quad \therefore dt = \frac{1}{u\cos\alpha - \mu x}. \quad = -\frac{1}{\mu}\cdot \log.(u\cos\alpha - \mu x)$

$$\therefore e^{-\mu t} = \frac{u\cos\alpha}{u\cos\alpha - \mu x}. \quad \therefore u\cos\alpha - \mu x = u\cos\alpha\, e^{-\mu t}.$$

$$\mu x = (u\cos\alpha)(1 - e^{-\mu t})$$

also $\frac{dy}{dt} = -gt - \mu y + u\sin\alpha.$

$$d_t\left(y e^{\frac{gt^2}{2}}\right) = e^{-\frac{gt^2}{2}} u\sin\alpha - \mu y \cdot e^{-\frac{gt^2}{2}}.$$

$$y \cdot e^{-\frac{gt^2}{2}} = -u\,t\sin\alpha \cdot e^{-\frac{gt^2}{2}} + u\sin\alpha.$$

$$\frac{dy}{dt} + \mu y = u\sin\alpha - gt.$$

$$d_t\left(y e^{\mu t}\right) = e^{\mu t} \cdot u\sin\alpha - e^{\mu t} \cdot gt.$$

$$\therefore y e^{\mu t} = \frac{1}{\mu} \cdot u\sin\alpha \cdot e^{\mu t} - g \cdot e^{\mu t}\left(\frac{t}{\mu} - \frac{1}{\mu^2}\right) + c.$$

$$\therefore \mu y = u\sin\alpha - gt + \frac{g}{\mu} + \mu C e^{-\mu t}.$$

$$0 = u\sin\alpha - \frac{g}{\mu}.$$

$$\therefore \mu y = u\sin\alpha - gt + \frac{g}{\mu}\left(1 - e^{-\mu t}\right) \quad \frac{g}{\mu} u\sin\alpha \cdot e^{-\mu t}.$$

$$\therefore \tan\beta = \frac{u\sin\alpha\, e^{-\mu t} - \frac{g}{\mu}\{1 - e^{-\mu t}\}}{u\cos\alpha\, e^{-\mu t}}$$

After the resisting medium

$$\frac{d^2y}{t^2} = -g. \qquad \frac{dx}{t} = 0.$$

$$\frac{dy}{t} = -gt + C. \qquad \frac{dx}{t'} = u\cos\alpha$$

$$= u\sin\alpha - gt.$$

$$\therefore \frac{dy}{x} = \frac{u\sin\alpha - gt_1}{u\cos\alpha} = \tan\beta.$$

$$\therefore \frac{u\sin\alpha - gt_1}{u\cos\alpha} = \frac{u\sin\alpha\, e^{\mu t} - \frac{g}{\mu}(1 - e^{-\mu t})}{u\cos\alpha\, e^{-\mu t}}$$

$$\therefore u\sin\alpha\, e^{-\mu t} - gt_1 e^{-\mu t} = u\sin\alpha\, e^{\mu t} - \frac{g}{\mu}\cdot 1 - e^{-\mu t}.$$

$$\frac{\mu}{g}\cdot t_1 = (e^{\mu t} - 1).$$

$$1 + \mu t_1 = e^{\mu t}.$$

216-14

Eqn. of path

$$y = x\tan\alpha - \frac{g x^2}{2u^2\cos^2\alpha}$$

$$t = \frac{u\sin\alpha}{g}$$

$$\therefore x = \frac{u^2\sin\alpha\cos\alpha}{g} = a$$

$$y = \frac{u^2\sin^2\alpha}{2g} = b \qquad (1)$$

$$x = (1-\varepsilon^{-\mu t})\frac{u\cos\alpha}{\mu}$$

$$y = \frac{u\sin\alpha}{\mu}(1-\varepsilon^{-\mu t}) + \frac{g}{\mu^2}(1-\varepsilon^{-\mu t}) - \frac{gt}{\mu}$$

At highest point $dy/dt = 0$

$$\therefore 0 = u\sin\alpha\cdot\varepsilon^{-\mu t} - g$$

$$dy/dt = 0 = \left(u\sin\alpha + \frac{g}{\mu}\right)\varepsilon^{-\mu t} - \frac{g}{\mu}$$

$$\therefore \varepsilon^{-\mu t} = 1 - \frac{\frac{g}{\mu}}{u\sin\alpha + \frac{g}{\mu}}$$

$$= \frac{u\sin\alpha}{u\sin\alpha + \frac{g}{\mu}} = \frac{\frac{\mu}{g}\cdot u\sin\alpha}{1 + \frac{\mu}{g}u\sin\alpha}$$

$$\therefore x = \frac{\frac{\mu}{g}\cdot u\sin\alpha}{1+\frac{\mu}{g}u\sin\alpha}\cdot\frac{u\cos\alpha}{\mu}$$

$$= \frac{u^2\sin\alpha\cos\alpha}{g}\left\{1 - \frac{\mu}{g}u\sin\alpha\right\}$$ neglecting μ^2

$$\therefore \quad \delta\alpha = x - \alpha = -\frac{\mu u^3 \sin^2\alpha \cos\alpha}{g^2}$$

$$= -\mu \cdot \left\{\frac{u^3 \tan\alpha}{g}\right\}^{\frac{1}{2}} \qquad \text{(eq (1)}$$

Again

$$y = \left\{\frac{u\sin\alpha}{\mu} + \frac{g}{\mu^2}\right\}\left\{1 - \varepsilon^{-\mu t}\right\} - \frac{gt}{\mu}$$

$$= \left\{\frac{u\sin\alpha}{\mu} + \frac{g}{\mu^2}\right\}\left\{\frac{\frac{\mu}{g}u\sin\alpha}{1+\frac{\mu}{g}u\sin\alpha}\right\} - \frac{gt}{\mu}$$

$$= \frac{g}{\mu^2}\left\{1 + \frac{\mu u\sin\alpha}{g}\right\}\frac{\left(\frac{\mu}{g}u\sin\alpha\right)}{1+\frac{\mu}{g}\cdot u\sin\alpha} - \frac{gt}{\mu}$$

$$= \frac{u\sin\alpha}{\mu} - \frac{gt}{\mu}$$

Now $\varepsilon^{\mu t} = 1 + \frac{\mu}{g}\cdot u\sin\alpha$

$$\therefore \quad \mu t = \text{Log}\cdot\left(1 + \frac{\mu}{g}\cdot u\sin\alpha\right)$$

$$\therefore y = \frac{u\sin\alpha}{\mu} - \frac{g}{\mu^2}\cdot \text{Log}\cdot\left(1 + \frac{\mu}{g}u\sin\alpha\right)$$

$$= \frac{u\sin\alpha}{\mu} - \frac{g}{\mu^2}\left\{\frac{\mu}{g}\cdot u\sin\alpha - \frac{1}{2}\cdot\frac{\mu^2}{g^2}\cdot u^2\sin^2\alpha + \frac{1}{3}\cdot\frac{\mu^3}{g^3}\cdot u^3\sin^3\alpha - \right\}$$

$$= \frac{1}{2}\cdot\frac{u^2\sin^2\alpha}{g} - \frac{1}{3}\cdot\frac{\mu}{g^2}\cdot u^3\sin^3\alpha$$

$$\therefore \delta\beta = y - \beta = -\frac{1}{3}\cdot\mu\cdot\frac{u^3\sin^3\alpha}{g^2}$$

$$= -\frac{1}{3}\cdot\mu\cdot\left(\frac{\rho\beta^3}{g}\right)^{\frac{1}{2}} \qquad \text{eq (1)}$$

Let $f(r)$ denote the force at a distance (r).

Then. $\frac{d^2x}{dt^2} = -f(r) \cdot \frac{x}{r}$.

$\therefore \left(\frac{dx}{dt}\right)^2 = -\int \frac{f(r)}{r} x^2 + C$.

when $x = a$. $r = a$

$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{f(a)}{a}$

at the beginning of the motion $x = a'$

$\therefore \frac{dx}{dt} = \frac{f(r)}{r} \cdot (a'^2 - x^2)$

$\therefore \frac{dt}{dx} = \sqrt{\frac{r}{f(r)}} \cdot \frac{1}{\sqrt{a'^2 - x^2}}$

$\therefore t = \sqrt{\frac{r}{f(r)}} \cos^{-1} \frac{x}{\sqrt{a'}} + C$.

when $t = 0$ $x = a'$ $\therefore$ $C = \sqrt{\frac{r}{fr}} \cdot 0 \therefore C = 0$

$\therefore t = \sqrt{\frac{r}{f(r)}} \cdot \cos^{-1} \frac{x}{\sqrt{a'}}$.

while body passes $r = a$ $x = dx = 0$.

$\therefore t = \sqrt{\frac{a}{f(a)}} \frac{\pi}{2}$.

A.B. the initial position of the particles
P. & R. the positions at a time t.
then the accelerating forces on P and R are respectively = $\frac{2\mu}{PA^2} \cdot \frac{CP}{PA}$

and $\frac{2\mu}{PA^2} \cdot \frac{CA}{PA}$.

wherefore they are respectively ∴ ∝ to PC & AC consequently the velocities with the particles are ∴ ∝ to PC and AC.
wherefore the particles will arrive simultaneously at C. and PA is always // to AB.

The part of the attraction on P // to $CP = \frac{2\mu}{CP^2} \cdot \sin\alpha$

∴ effective attraction P // to $PA = \frac{2\mu}{CP^2} \sin^2\alpha$.

similarly ——— A // to $PA = \frac{\mu}{CP^2} \cos^2\alpha$.

∴ whole attraction // to $PA = \frac{\mu}{CP^2} = \frac{\mu}{x^2}$ suppose.

$$\therefore \frac{d^2x}{dt^2} = -\frac{\mu}{x^2}.$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = 2\mu \cdot \left(\frac{1}{x} - \frac{1}{a}\right).$$

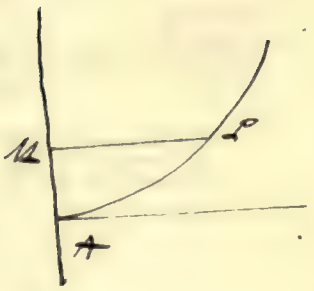
$$\frac{dt}{dx} = \frac{\sqrt{a}}{\sqrt{2\mu}} \cdot \frac{x}{\sqrt{ax - x^2}}$$

$$\therefore t = \frac{1}{2}\sqrt{\frac{a}{2\mu}} \cdot \left\{ -\sqrt{ax - x^2} + a\pi - a\cos^{-1}\frac{2x}{a} \right\}.$$

at C $x = 0$.

$$\therefore t = \frac{\pi}{\sqrt{\mu}} \cdot \left(\frac{a}{2}\right)^{\frac{3}{2}}.$$

Let L be the initial position of the particle.



Then $\left(\frac{ds}{dt}\right)^2 = 2fx$.

or $\left(\frac{ds}{dt}\right)^2 = 2gx + C$.

Let x initially $= h$ and velocity $= 0$.

$$\left(\frac{ds}{dt}\right)^2 = 2g(h - x). \quad (1)$$

From the common parabola curve we have

$$\frac{dy}{dx} = \frac{\sqrt{2a-x}}{\sqrt{x}} \quad \text{or} \quad x(dy)^2 = (2a-x)(dx)^2.$$

$$\therefore x(ds)^2 = 2a.(dx)^2.$$

$$\therefore (ds)^2 = \left(\frac{2a}{x}\right) dx^2.$$

$$\therefore \text{from (1)}. \left(\frac{dx}{dt}\right)^2 = \frac{2g}{2a}.(hx - x^2).$$

$$\therefore \frac{dt}{dx} = \sqrt{\frac{a}{g}} \cdot \frac{1}{\sqrt{hx - x^2}}.$$

$$\therefore t = \sqrt{\frac{a}{g}}\left(\sin^{-1}\frac{2x}{h} + C.\right)$$

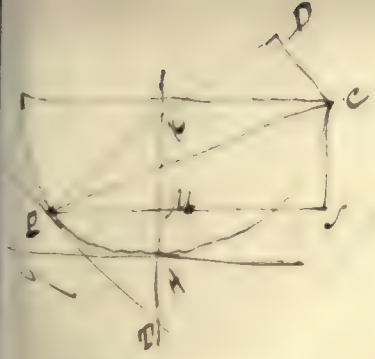
when $t = 0 \quad x = h$

$$\therefore T = \sqrt{\frac{a}{g}}\left\{\sin^{-1}\frac{2x}{h} + \pi\right\}$$

when the particle arrives at the lowest point $x = 0$

$$\therefore T' = \pi\sqrt{\frac{a}{g}}.$$

CP



$AM = x$. $ML = y$ the co-ords of L.

Now $\tan LTM = \tan DLM = \frac{dy}{dx} = \frac{\sqrt{2ax-x^2}}{x}$

$$\tan CLM = \frac{CS}{LS} = \frac{2a-x}{a(\pi + \sin^{-1}\frac{x}{a}) + \sqrt{2ax-x^2}}$$

$$\therefore \tan DLC = \frac{\frac{\sqrt{2ax-x^2}}{x} - \frac{(2a-x)}{a(\pi+\sin^{-1}\frac{x}{a}) + \sqrt{2ax-x^2}}}{1 + \frac{(2a-x)\sqrt{2ax-x^2}}{x\{a(\pi+\sin^{-1}\frac{x}{a}) + \sqrt{2ax-x^2}}}$$

$$= \frac{a(\pi+\sin^{-1}\frac{x}{a})\sqrt{2ax-x^2}}{x(a(\pi+\sin^{-1}\frac{x}{a}) + 2x\sqrt{2ax-x^2}}$$

Also $CD = CL \sin CLD$.

$$\sin CLD = \frac{\tan CLD}{\sqrt{1+\tan^2 CLD}}$$

$$= \frac{x(\pi+\sin^{-1}\frac{x}{a})\sqrt{2ax-x^2}}{\sqrt{(\pi+\sin^{-1}\frac{x}{a})^2 2ax + 2x(\pi+\sin^{-1}\frac{x}{a})\sqrt{2ax-x^2} + 4x^2(2ax-x^2)}}$$

Now acceleration force of gravity down $CL = -g\frac{CS}{CL}$.

$\therefore$ for the motion down the chord CL

$$\frac{d^2x}{dt^2} = +g\frac{CS}{CL}.$$

$$\therefore \frac{dx}{dt} = +gt\cdot\frac{CS}{CL}. \quad ; C' = 0.$$

$$x = +\frac{1}{2}gt^2\cdot\frac{CS}{CL} + C'$$

$x = 2a$ when $t = t'$ $\therefore C = 0$

$\therefore$ [illegible]

$\therefore x = 2a - \frac{1}{2}g t^2 \frac{CS}{CL}$.

At L, $x = CL$ $\therefore CL = \frac{1}{2}g t^2 \cdot \frac{CS}{CL}$ $\therefore t = CL\cdot\sqrt{\frac{2}{g\cdot CS}}$

data for the motion down the arc.

$$\frac{d^2 s}{t'^2} = -g \frac{dx}{s}$$

$$\left(\frac{ds}{t'}\right)^2 = -2g(x - 2a) \quad \text{for } x = 2a \text{ write } \frac{ds}{t} = 0.$$

$$\therefore \frac{dt'}{s} = \frac{1}{\sqrt{2g}} \cdot \frac{1}{\sqrt{2a - x}} \cdot \quad \therefore \frac{dt'}{x} = \sqrt{\frac{a}{g}} \cdot \frac{1}{\sqrt{2ax - x^2}}.$$

$$t' = \sqrt{\frac{a}{g}} \left\{ \pi + \text{vers}^{-1} \frac{x}{a} \right\} - c\sqrt{(2ax - x^2)},$$

$$\therefore \frac{t}{t'} = \frac{\sqrt{\frac{a}{g}} \left\{ \pi + \text{vers}^{-1} \frac{x}{a} \right\}}{\sqrt{\frac{2}{g(2a-x)}} \cdot cP} = \sqrt{\frac{a(2a-x)}{2}} \cdot \left(\frac{\pi + \text{vers}^{-1} \frac{x}{a}}{cP} \right\}.$$

$$\therefore t' : t :: cP : \left(\pi + \text{vers}^{-1} \frac{x}{a}\right) \sqrt{\frac{2a-x}{2a}} \cdot (ct).$$

Now $cP : \sqrt{sP^2 + sc^2} = \sqrt{a^2\left(\pi + \text{vers}^{-1}\frac{x}{a}\right)^2 + 2a\left(\pi + \text{vers}^{-1}\frac{x}{a}\right) + 2ax - x^2 + (2a-x)^2}$

$$= \sqrt{\frac{a}{2x}} \sqrt{2ax\left(\pi + \text{vers}^{-1}\frac{x}{a}\right)^2 + 4x\left(\pi + \text{vers}^{-1}\frac{x}{a}\right)\sqrt{2ax - x^2} + 4x(2a-x)}$$

and from above $cD = cP \sin cPD$

$$= \sqrt{\frac{a}{2x}} \left\{ \pi + \text{vers}^{-1}\frac{x}{a} \right) \sqrt{2ax - x^2}$$

$$= \sqrt{\frac{a(2a-x)}{2}} \cdot \left(\pi + \text{vers}^{-1}\frac{x}{a}\right)$$

wherefore from A.

$$t' : t :: cP : cD.$$

$AA' - BA\cos\theta = \theta$.

$\therefore \angle BAC = \frac{\pi}{2} - \theta$.

accelerating force down the ~~arc~~ chord $= -g\sin\left(\frac{\pi}{2}-\theta\right)$

$\therefore \frac{d^2x}{dt^2} = +g\cos\left(\frac{\pi}{2}-\theta\right)$.

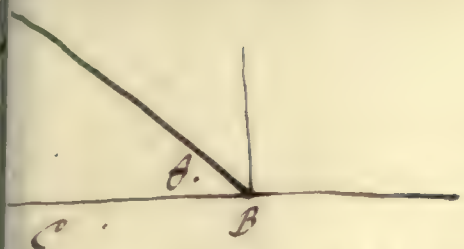
$x = \frac{1}{2}gt^2\cdot\cos\left(\frac{\pi}{2}-\theta\right)$.

$\therefore$ Vel. $v^2 = a^2\cos^2\theta = a^2\cdot\sin\left(\frac{\pi}{2}-2\theta\right)$

$= 2a^2\sin\left(\frac{\pi}{2}-\theta\right)\cos\left(\frac{\pi}{2}-\theta\right)$.

$\therefore 2a^2\sin\left(\frac{\pi}{2}-\theta\right)\cos\left(\frac{\pi}{2}-\theta\right) = \frac{1}{4}g^2t^4\cos^2\left(\frac{\pi}{2}-\theta\right)$

$\text{or } T = \left(8\cdot\frac{a}{g}\right)^{\frac{1}{2}}\cdot\sin\left(\frac{\pi}{2}-\theta\right)^{-\frac{1}{4}}$.



for the portion above the plane.

$\frac{AB}{C} = g \sin\theta$.

$\therefore \left(\frac{dl}{t}\right) = gt \sin\theta$. = velocity at B in direct. ⊥ B.

$\therefore$ vel. ∥ to $CB = gt \sin\theta \cos\theta$. vel. ⊥ to $CB = gt \sin^2\theta$.
$= A \sin\theta\cos\theta$ $= A\sin^2\theta$.

$\therefore$ the greatest height the body reaches $\frac{A^2 \sin^4\theta}{2g}$.

$\therefore$ the time it requires to reach this height $= s = \frac{1}{2}gt^2$

$\therefore t^2 = A^2 \sin^4\theta$ $\frac{2s}{g} = t^2$.

$t = \frac{A \sin^2\theta}{g}$.

$\therefore$ time of flight $= \frac{2A}{g} \sin^2\theta$

horizontal velocity $= A \sin\theta\cos\theta$

$\therefore R = \frac{2A^2}{g} \sin^2\theta \cos\theta$. which must be a maximum.

$\therefore \frac{dR}{d\theta} = 3 \sin^2\theta \cos^2\theta - \sin^4\theta$.

$= 3 \sin^2\theta (1 - \sin^2\theta) - \sin^4\theta$

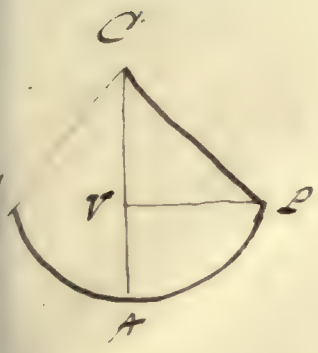
$= 3\sin^2\theta - 4\sin^4\theta$.

$\therefore \sin^4\theta - \frac{3}{4}\sin^2\theta + \frac{9}{64} = \frac{9}{64}$.

$\sin^2\theta - \frac{3}{8} = \pm \frac{3}{8}$.

$\sin^2\theta = \frac{6}{8} = \frac{3}{4}$

$\therefore \sin\theta = \frac{1}{2}\sqrt{3}$.



The square of the velocity at the lowest point of the curve $= 2g.VA = 2ga$.

∴ the velocity^2 of each particle after rebounding $= e^2 2ga$.

∴ if x, y be the coords. of the point in the curve through which the particles will rise, $x = \frac{e^2 2ga}{2g} = \frac{a}{2}.e^2$

The chord of this arc $= \sqrt{x^2+y^2} = \sqrt{2ax}$ for $y^2 = 2ax - x^2$

$= \sqrt{2a}\left\{\sqrt{\frac{a}{2}}.e\right\} = ae$.

similarly the chord of the arc described after the second rebound $= ae^2$

$\therefore S = a(1 + e + e^2 + \text{&c.})$

$= \frac{a}{1-e}$.

v, v' the velocities of B. A. after contact.

$\therefore v + v' = eu$ & $A(u' + v) = Bv'$ $\therefore v' = \frac{A(u'+v)}{B} = eu - v$

$$\therefore v = -\left(\frac{A + B(e)'}{A+B}\right) u' \quad v' = \frac{A + A(e)'}{A+B} u'.$$

$$\text{or } v = \left(\frac{A - B(-e)'}{A+B}\right) u' \quad ; \quad v' = \frac{A - A(-e)'}{A+B} u' \quad (3).$$

Let v_1, v_1' be the velocities of A. B after the second collision (which will evidently take place at the lowest point of the cycloid since the curve is tautochronous.)

$$\therefore v_1' - v_1 = e(v' + v) \quad ; \quad A(v - v_1) = B(v' + v_1')$$

$$\therefore v_1' = ev' + ev + v_1 = \frac{Av - Av_1 - Bv'}{B}$$

$$\therefore v_1 = -\left(\frac{B(e+1)v' + (Be - A)v}{A+B}\right) u'$$

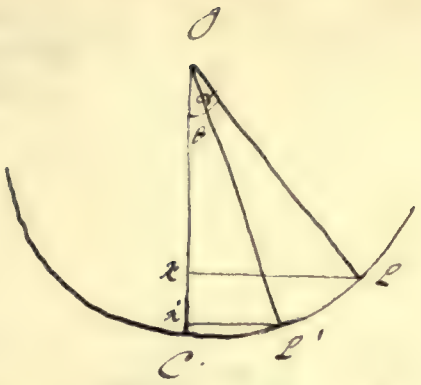
$$v_1' = \left(\frac{Ae(v'+v) - Bv' + Av}{A+B.}\right) u'$$

substituting the values of v, v' from (3)

$$v_1 = -\frac{A + Be^2}{A+B} = \frac{A + B(-e)^2}{A+B.} \quad \text{(which is negative)}$$

$$v_1' = \frac{A - Ae^2}{A+B} = \frac{A - A(-e)^2}{A+B.}$$

and similarly might be found the velocity after x impacts.



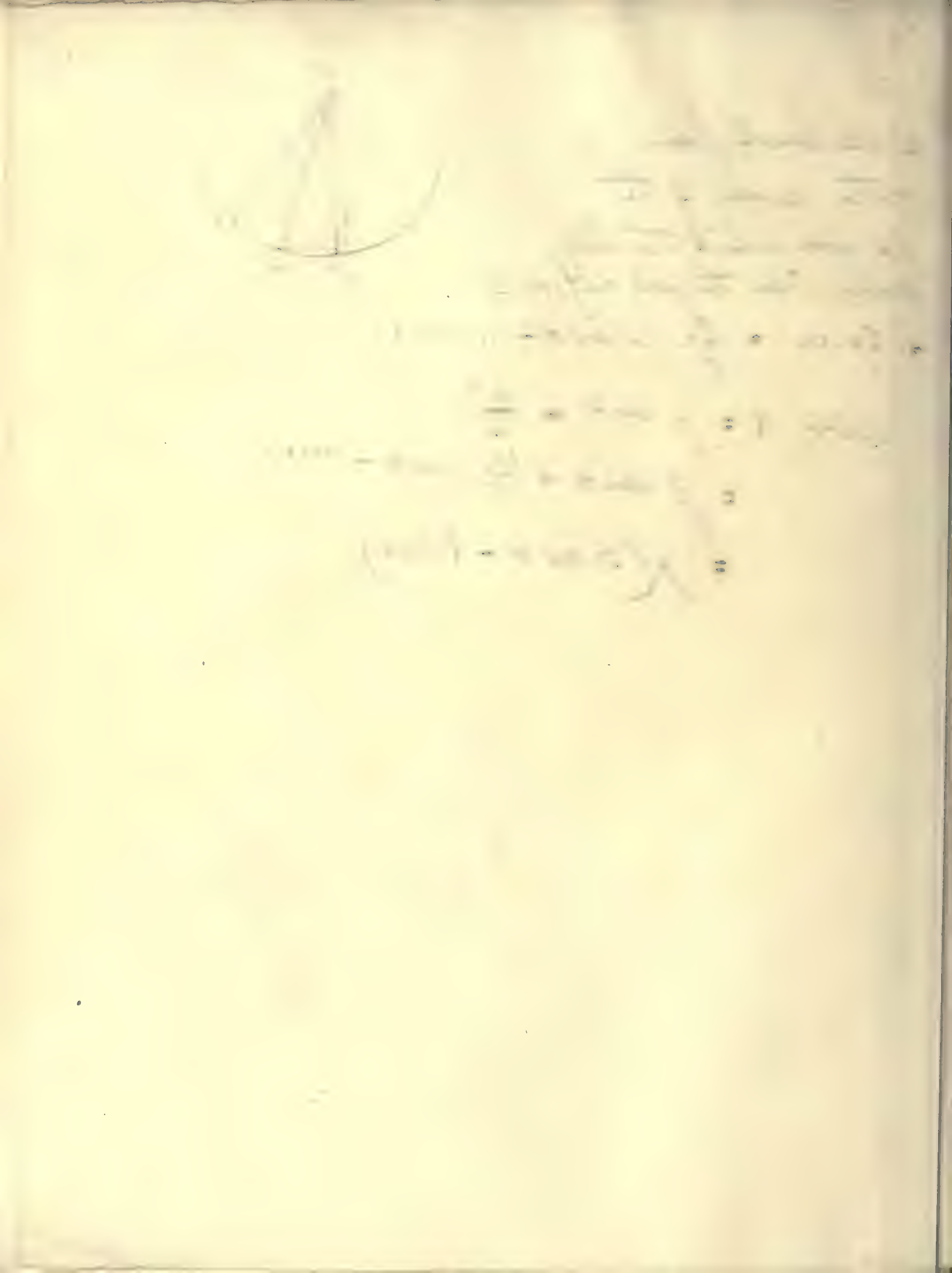
C° the lowest point of the curve. Q the pt from which the body moves. Then the (velocity)^2 at Q'

$= 2g.xx' = 2g.(a\cos\theta - a\cos\alpha)$

also $R = g\cos\theta + \frac{v^2}{a}$.

$= g\cos\theta + 2g(\cos\theta - \cos\alpha)$

$= g(3\cos\theta - 2\cos\alpha)$.



The ^{n} falls curve is $y = \log_{\alpha} x$.

Now $R = g \frac{dy}{ds} - \frac{v^2}{\rho}$.

But $v^2 = 2g(x-h)$

also $\frac{ds}{dx} = \sqrt{1+(dy)^2} = \frac{\sqrt{A^2x^2+1}}{Ax}$ where $A = \log \alpha$

$\frac{dy}{ds} = \frac{dx}{ds}\frac{dy}{dx} = \frac{Ax}{\sqrt{A^2x^2+1}}$. $\frac{1}{Ax} = \frac{1}{\sqrt{A^2x^2+1}}$.

also $\rho = -\frac{1}{\frac{d^2y}{dx^2}}\left(\frac{ds}{dx}\right)^{\frac{3}{2}} =$

$= \frac{Ax^2}{A^3x^3}\left\{\frac{A^2x^2+1}{A^2x^2}\right\}^{\frac{3}{2}} = \frac{(A^2x^2+1)^{\frac{3}{2}}}{A^2x}$.

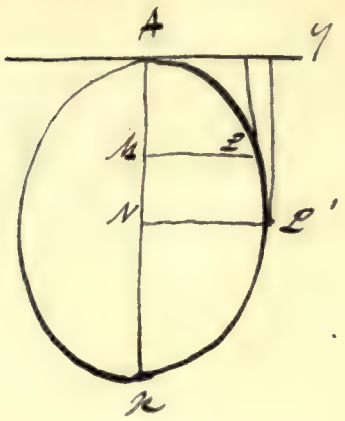
wherefore $R = \frac{g}{\sqrt{A^2x^2+1}} - \frac{2g(x-h)A^2x}{(A^2x^2+1)^{\frac{3}{2}}} = 0$

$A^2x^2+1-2(x-h)A^2x = 0$

$A^2(x^2-2hx+h^2) = 1+A^2h^2$

$x = \frac{1}{A}\left\{Ah + \sqrt{1+A^2h^2}\right\}$

X



Adopting the notation we have

$$R = g\frac{dy}{ds} - \frac{v^2}{\rho} = 0.$$

The velocity at $P' = 2g(x-h)$.

$$\frac{dy}{dx} = \frac{b}{a}\cdot\frac{a-x}{\sqrt{2ax-x^2}}, \quad \frac{ds}{dx} = \frac{\sqrt{a^2b^2+(a^2-b^2)(2ax-x^2)}}{a\sqrt{2ax-x^2}}.$$

$$\therefore \frac{dy}{ds} = \frac{b(a-x)}{\sqrt{a^2b^2+(a^2-b^2)(2ax-x^2)}}$$

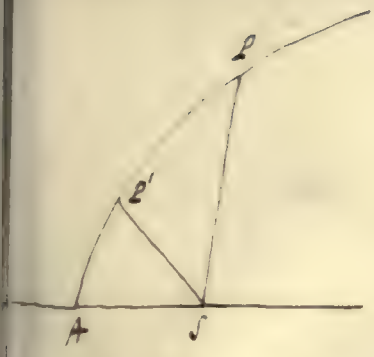
$$\frac{d\rho}{dx} = \frac{(2ax-x^2)^{\frac{3}{2}}}{ab}\left\{\frac{a^2b^2+(a^2-b^2)(2ax-x^2)}{a^2(2ax-x^2)}\right\}^{\frac{3}{2}}$$

$$= \frac{\left\{a^2b^2+(a^2-b^2)(2ax-x^2)\right\}^{\frac{3}{2}}}{a^4b}$$

$$\therefore R = g\cdot\frac{b(a-x)}{\sqrt{a^2b^2+(a^2-b^2)(2ax-x^2)}} - \frac{2g(x-h)a^4b}{\left(a^2b^2+(a^2-b^2)(2ax-x^2)\right)^{\frac{3}{2}}} = 0.$$

or $b(a-x)\left\{a^2b^2+(a^2-b^2)(2ax-x^2)\right\} - 2(x-h)a^4b = 0.$

$$(a^2-b^2)\left\{x^3-3ax^2\right) - 3b^2a^2x + a^2(b^2+2ah) = 0.$$



Accel. $R = \frac{\mu k}{r^3} - \frac{v^2}{\rho}$.

Also $\frac{d^2 s}{dt^2} = -\frac{\mu}{r^2} \frac{dr}{dt}$.

$\therefore 2 \frac{ds}{dt} \frac{d^2 s}{dt^2} = -\frac{2\mu}{r^2} \cdot \frac{dr}{dt}$.

$\left(\frac{ds}{dt}\right)^2 = 2\left(\frac{\mu}{r} - \frac{\mu}{a}\right)$

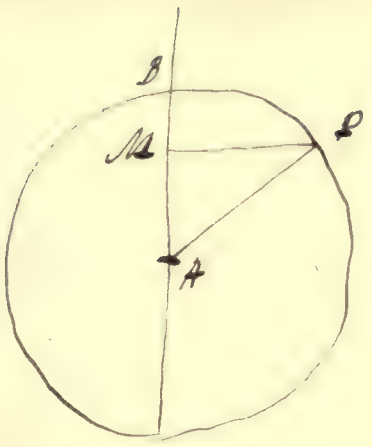
or $v^2 = \beta^2 k \cdot \left(\frac{\mu}{r} - \frac{\mu}{a}\right)$.

Also $\rho = 2 \cdot \frac{r^{\frac{3}{2}}}{m^{\frac{1}{2}}}$ $\therefore \frac{v^2}{\rho} = \left(\frac{m}{r^3}\right)^{\frac{1}{2}} \left\{\left(\frac{\mu}{r} - \frac{\mu}{a}\right) + \frac{\beta^2}{2}\right\}$.

Also $\beta = \sqrt{mr}$ $\therefore \frac{\mu k}{r^3} = \frac{\mu}{r} \cdot \left(\frac{m}{r^3}\right)^{\frac{1}{2}}$.

$\therefore R = \left(\frac{m}{r^3}\right)^{\frac{1}{2}} \left\{\frac{\mu}{r} - \left(\frac{\mu}{r} - \frac{\mu}{a}\right) - \frac{\beta^2}{2}\right\}$.

$= \left(\frac{m}{r^3}\right)^{\frac{1}{2}} \left\{\frac{\mu}{a} - \frac{\beta^2}{2}\right\}$.



Acceln. $R = -g\frac{dy}{ds} + \frac{v^2}{\rho}$.

Prin. $\frac{d^2s}{dt^2} = +g\frac{dx}{ds}$

$\therefore \left(\frac{ds}{dt}\right)^2 = 2gx + C$.

at B $x=0$. and $\left(\frac{ds}{dt}\right)^2 = \beta^2$.

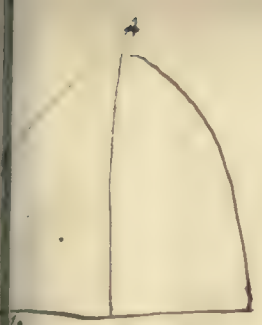
$\therefore \left(\frac{ds}{dt}\right)^2 = \beta^2 + 2gx = \beta^2 + 2g\{a - AM\}$.

also $\frac{dy}{ds} = \frac{dy}{dx}\cdot\frac{dx}{ds} = \frac{a-x}{\sqrt{2ax-x^2}} \cdot \frac{\sqrt{2ax-x^2}}{a}$

$= \frac{a-x}{a} = \frac{AM}{a}$

$\therefore R = -g\frac{AM}{a} + \frac{1}{a}\{\beta^2 + 2g(a - AM)\}$.

$= \frac{\beta^2}{a} + g\{2 - 3\cos\theta\}$. $\because B\hat{A}P = \theta$



Let R be the resultant force.

$$R = -g\frac{dy}{ds} - \frac{v^2}{\rho}.$$

But if h = space due to velocity of projection

$$v^2 = 2g(h-x). \quad ; x \text{ \& } y \text{ being co-ordinates}$$

of the point at which the body leaves the curve.

$$\therefore \text{at this point } 0 = -g\frac{dx}{ds} + \frac{2g(h-x)}{\rho}.$$

$$\rho = -g\cdot\left(\frac{ds}{dx}\right)^2 - 2(h-x)\frac{d^2y}{dx^2}g. \quad \#.$$

= an \# gives the value of x for the point at which the particle leaves the curve. Let this $= x'$

For the direction of the particle when

$$\frac{dy}{dx} = \frac{a-x'}{\sqrt{ax'-x'^2}} = \text{tan of the angle} = \beta.$$

which the direct. of the particle makes with the horizon at the time of leaving the curve.

also the space due to velocity at this point $= 2(h-x')$

$$\therefore y = x\beta - \frac{g\alpha^2}{4g(h-x')}\left\{1+\beta^2\right\} \text{ is the } = \text{n locus of the path of the}$$

particle on leaving the curve.

At the $y = \beta$. $x = b - x'$.

$$\therefore \beta = (b-x')\beta - \frac{(b-x')^2}{4\cdot(h-x')}\cdot(1+\beta^2) \text{ which determines}$$

h and consequently the velocity of projection.

Let α be the angle of inclination.

Then the motion is determined by $\ddot{x} = a$.

$$\ddot{x} = +g \sin\alpha.$$

$$\therefore \dot{x}_t = gt \sin\alpha. \quad = \text{velocity down the plane.}$$

at time t.

The path of this estimated in a horizontal direction is $gt \sin\alpha \cos\alpha$ which is to be a max.

$$\therefore u = gt \sin\alpha \cos\alpha.$$

$$\dot{u}_1 = gt(-\sin^2\alpha + \cos^2\alpha)$$

$$\therefore 1 - 2\sin^2\alpha = 0.$$

$$\alpha = \sin^{-1}\frac{1}{\sqrt{2}} = 45^\circ.$$

(4)

$$u = \frac{1 - \sin x + \cos x}{\sin x + \cos x - 1} = \frac{0}{0} \quad \text{for } x = \frac{\pi}{2}.$$

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}' = \frac{-\cos x - \sin x}{\cos x - \sin x} = 1 \quad \text{for } x = \frac{\pi}{2}.$$

---

(5)

$$u = \frac{a - x - a \log a + a \log x}{a - \sqrt{2ax - x^2}} = \frac{0}{0} \quad \text{for } x = a.$$

$$= \frac{a - a - h - a \log a + a \log (a+h)}{a - \sqrt{a^2 - h^2}} \quad \text{for } x = a + h.$$

$$= \frac{-h - a \log a + a \left\{ \log a + \frac{h}{a} - \frac{h^2}{2a^2} \alpha \right\}}{a - \left\{ a - \frac{1}{2} \frac{h^2}{a} + \alpha \right\}}.$$

$$= \frac{-\frac{h^2}{2a}}{\frac{h^2}{2a}} = -1$$

Ende p. 93.

$$u = \frac{1}{2x^2} - \frac{\pi}{2x \cdot 2\pi x} = \infty - \infty \text{ of } x = 0.$$

$$u = \frac{2x \cdot \tan \pi x - 2\pi \cdot x^2}{4x^3 \tan \pi x \cdot} = \frac{0}{0} \text{ of } x = 0.$$

$$= \frac{x \tan \pi x - \pi x^2}{2x^3 \tan \pi x \cdot} = \frac{\tan \pi x + \pi x \cdot \sec^2 \pi x - 2\pi x}{6x^2 \tan \pi x + 2\pi x^3 \sec^2 \pi x \cdot} \frac{0}{0}.$$

$$= \frac{\pi \cdot \sec^2 \pi x + \pi \sec^2 \pi x + \pi^2 x \cdot 2 \sec^2 \pi x \tan \pi x - 2\pi}{12x \cdot \tan \pi x \cdot + 6\pi x^2 \cdot \sec^2 \pi x \cdot + 6\pi x^2 \sec^2 \pi x + 4x^3 \pi^2 \sec^2 \pi x \tan \pi x.}$$

$$= \frac{4\pi^2 \sec^2 \pi x \tan \pi x + 2\pi^2 \sec^2 \pi x \tan \pi x + A'x}{12 \tan \pi x + 12\pi x \sec^2 \pi x + Ax} = \frac{0}{0} \text{ of } x = 0.$$

$$= \frac{6\pi^2 \sec^2 \pi x}{12} \text{ of } x = 0. = \frac{\pi^2}{2}.$$

$$u = \frac{e^x - e^{\sin x}}{x - \sin x} = \frac{x - \sin x + \frac{1}{2}(x^2 - \sin^2 x) + \frac{1}{6}(x^3 - \sin^3 x) \cdots}{x - \sin x} = \frac{0}{0} \text{ for } x = 0.$$

$$\frac{du}{dx} = \frac{1 - \cos x + x - \sin x \cos x + \frac{1}{2}(x^2 - \sin^2 x \cos x) + \frac{1}{6}(x^3 - \sin^3 x \cos x)}{1 - \cos x} = \frac{0}{0}.$$

$$\frac{d^2u}{dx^2} = \frac{\sin x + 1 + \sin^2 x - \cos^2 x + x - \sin x \cos^2 x + \frac{\sin^3 x}{2} + \frac{1}{2}x^2 - \frac{1}{2}\sin^2 x \cos^2 x + \frac{x^2}{4}}{\sin x} = \frac{0}{0} \text{ for } x = 0.$$

$$\frac{d^3u}{dx^3} = \frac{\cos x + 4 \sin x \cos x + 1 - \cos^3 x + 2\cos x \sin^2 x + \cdots}{\cos x} = 1 \text{ for } x = 0$$

$$\therefore \frac{e^x - e^{\sin x}}{x - \sin x} = 1 \text{ for } x = 0.$$

Hall p. 13.

$$u = \frac{\tan x - \sin x}{\sin^3 x} = \frac{0}{0} \text{ if } x = 0$$

$$= \frac{\sec^2 x - \cos x}{3 \sin^2 x \cos x} = \frac{0}{0} \text{ if } x = 0.$$

$$= \frac{2 \sec^2 x \cdot \tan x + \sin x}{6 \sin x \cos^2 x - 3 \sin^3 x} = \frac{0}{0} \text{ if } x = 0.$$

$$= \frac{4 \sec^2 x \cdot \tan^2 x + 2 \sec^4 x + \cos x}{6 \cos^3 x + 12 \cos x \sin^2 x - 9 \sin^2 x \cos x} = \frac{3}{6} = \frac{1}{2}$$

if $x = 0$.

$$\therefore u = \frac{\tan x - \sin x}{\sin^3 x} = \frac{1}{2} \text{ if } x = 0.$$

(15)

$$u = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2} = \frac{0}{0} \text{ if } x = 1.$$

$$= \frac{-n(n+1)x^{n-1} + n(n+1)x^n}{-2(1-x)} = \frac{0}{0} \text{ if } x = 1$$

$$= \frac{n(n+1)\left\{ nx^{n-1} - (n-1)x^{n-2} \right\}}{2} = \frac{1}{2} n(n+1) \text{ if } x = 1.$$

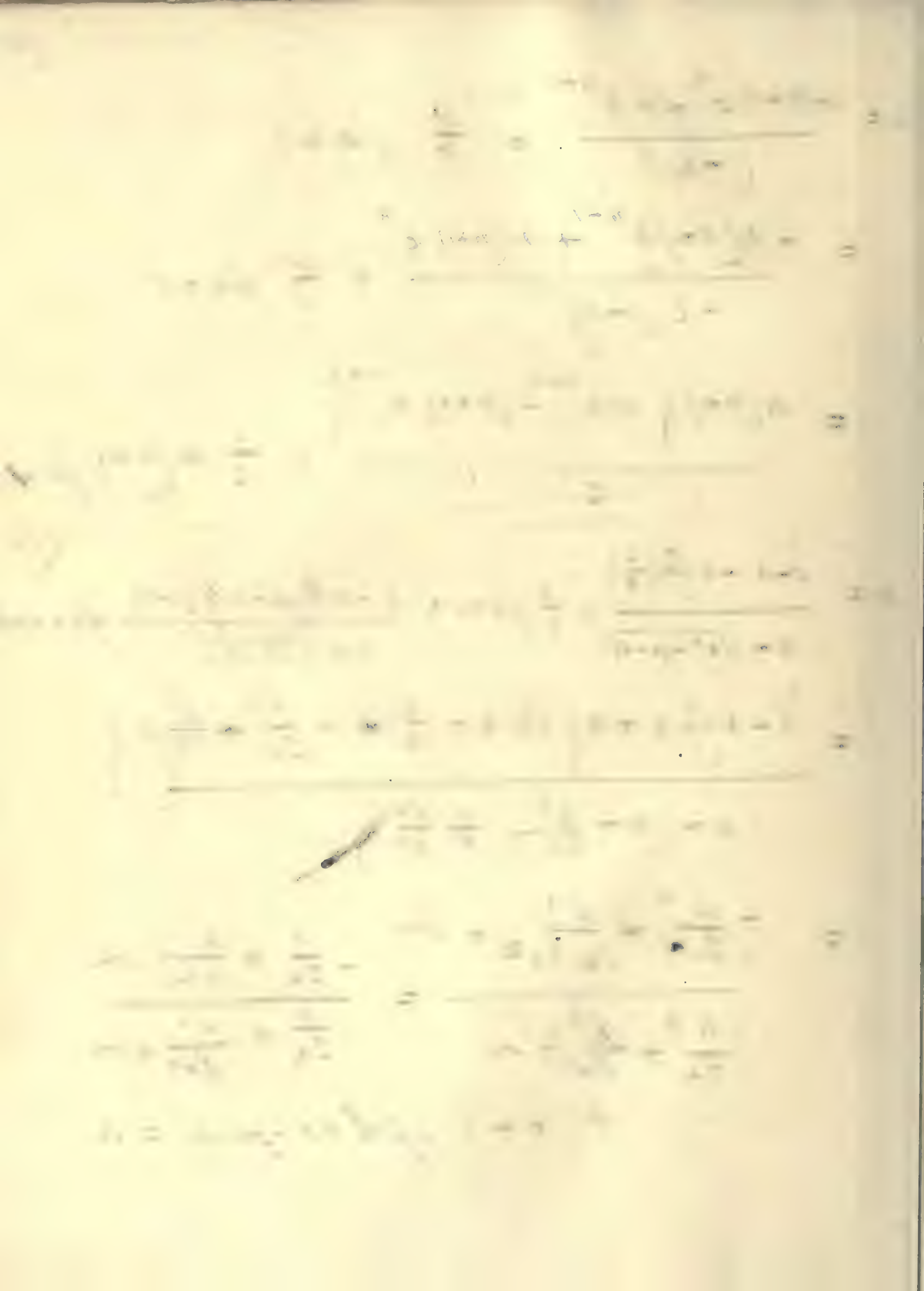
(16)

$$u = \frac{a - x - a\log\left(\frac{a}{x}\right)}{a - \sqrt{a^2 - (a-x)^2}} = \frac{0}{0} \text{ if } x = a = \frac{h - a\log a + a\log(a-h)}{a - \sqrt{a^2 - h^2}} \text{ if } x = a+h.$$

$$= \frac{h - a\log a + a\left\{ \log a - \frac{h}{a} - \frac{h^2}{2a^2} + \frac{h^3}{1.2.3} \text{ etc.} \right\}}{a - \left(a - \frac{h^2}{2a} - \frac{1}{8}\frac{h^4}{a^2} \text{ etc.}\right)}.$$

$$= \frac{-\frac{h^2}{2a} + \frac{h^3}{1.4.3a^2} + \text{etc.}}{\frac{h^2}{2a} + \frac{h^4}{8a^2} + \text{etc.}} = \frac{-\frac{1}{2a} + \frac{h}{1.4.3a^2} \text{ etc.}}{\frac{1}{2a} + \frac{h^2}{8a^2} + \text{etc.}}$$

$$= -1 \text{ if } h = 0 \text{ or } x = a.$$



If $\cos(m) = \cos(m+y) + x$. Expand y in terms of x.

since $\cos(m+y) = \cos(m) - x$

$$\text{whence } m+y = \cos^{-1}(\cos(m) - x)$$

$$y = \cos^{-1}(\cos(m) - x) - m.$$

$$= 0 \text{ if } x = 0$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\cos^2 m - 2x\cos m + x^2}} = \frac{1}{\sqrt{1-\cos^2 m}} \text{ if } x = 0$$

$$= \frac{1}{\sin m}.$$

$$\frac{d^2y}{dx^2} = \frac{-\cos m + x}{(1-\cos^2 m + 2x\cos m - x^2)^{\frac{3}{2}}}$$

$$= -\frac{\cos m}{(1-\cos^2 m)^{\frac{3}{2}}} \text{ if } x = 0$$

$$= -\cos m \cdot \frac{1}{\sin^2 m}.$$

$$\text{Taylor } y = y_{x=0} + \frac{dy}{dx}_{x=0} x + \frac{d^2y}{dx^2}_{x=0} \frac{x^2}{1\cdot 2} + \&c.$$

$$= \frac{x}{\sin m} - \frac{1}{2}\left(\frac{x^2}{\sin m}\right) \&c.$$

Hall p. 54.

$\sin u = m \sin x$; Expand u in a series ascending by powers of x.

$$u_{x=0} = \sin^{-1}(m \sin x) = 0 \text{ if } x = 0.$$

$$\frac{du}{dx} = \frac{m \cos x}{\sqrt{1 - m^2 \sin^2 x}} = m \text{ if } x = 0$$

$$\frac{d^2u}{dx^2} = \left(-m\sqrt{1 - m^2 \sin^2 x} \cdot \sin x + \frac{m^3 \cos^2 x \sin x}{\sqrt{1 - m^2 \sin^2 x}} \right) \Big| \frac{1}{1 - m^2 \sin^2 x}$$

$$= \frac{\left\{ m^3 (1 - \cos^2 x) - m + m^3 \cos^2 x \right\} \sin x}{(1 - m^2 \sin^2 x)^{\frac{3}{2}}}$$

$$= \frac{(m^3 - m) \sin x}{(1 - m^2 \sin^2 x)^{\frac{3}{2}}} = 0 \text{ if } x = 0.$$

$$\frac{d^3u}{dx^3} = \frac{(1 - m^2 \sin^2 x)^{\frac{3}{2}} (m^3 - m) \cos x + (m^5 - m^3) \sin x \, 3 (1 - m^2 \sin^2 x)^{\frac{1}{2}} \sin x \cos x}{(1 - m^2 \sin^2 x)^3}$$

$$= \frac{(1 - m^2 \sin^2 x)(m^3 - m) \cos x + m^2 (m^3 - m) 3 \sin^2 x \cos x}{(1 - m^2 \sin^2 x)^{\frac{5}{2}}} = \frac{(1 - 2m^2 \sin^2 x)(m^3 - m) \cos x}{(1 - m^2 \sin^2 x)^{\frac{5}{2}}}.$$

$$= m^3 - m \text{ if } x = 0.$$

$$\therefore u = u_{x=0} + \frac{du}{dx}_{x=0} x + \frac{d^2u}{dx^2}_{x=0} \frac{x^2}{12} + \text{—}$$

$$= mx + m(m^2 - 1) \frac{x^3}{123} + \alpha.$$

Leaf 57.

$$\frac{1}{x^2\sqrt{a+bx+cx^2}} = \frac{1}{x^3\left(ax^{-2}+bx^{-1}+c\right)^{\frac{1}{2}}}.$$

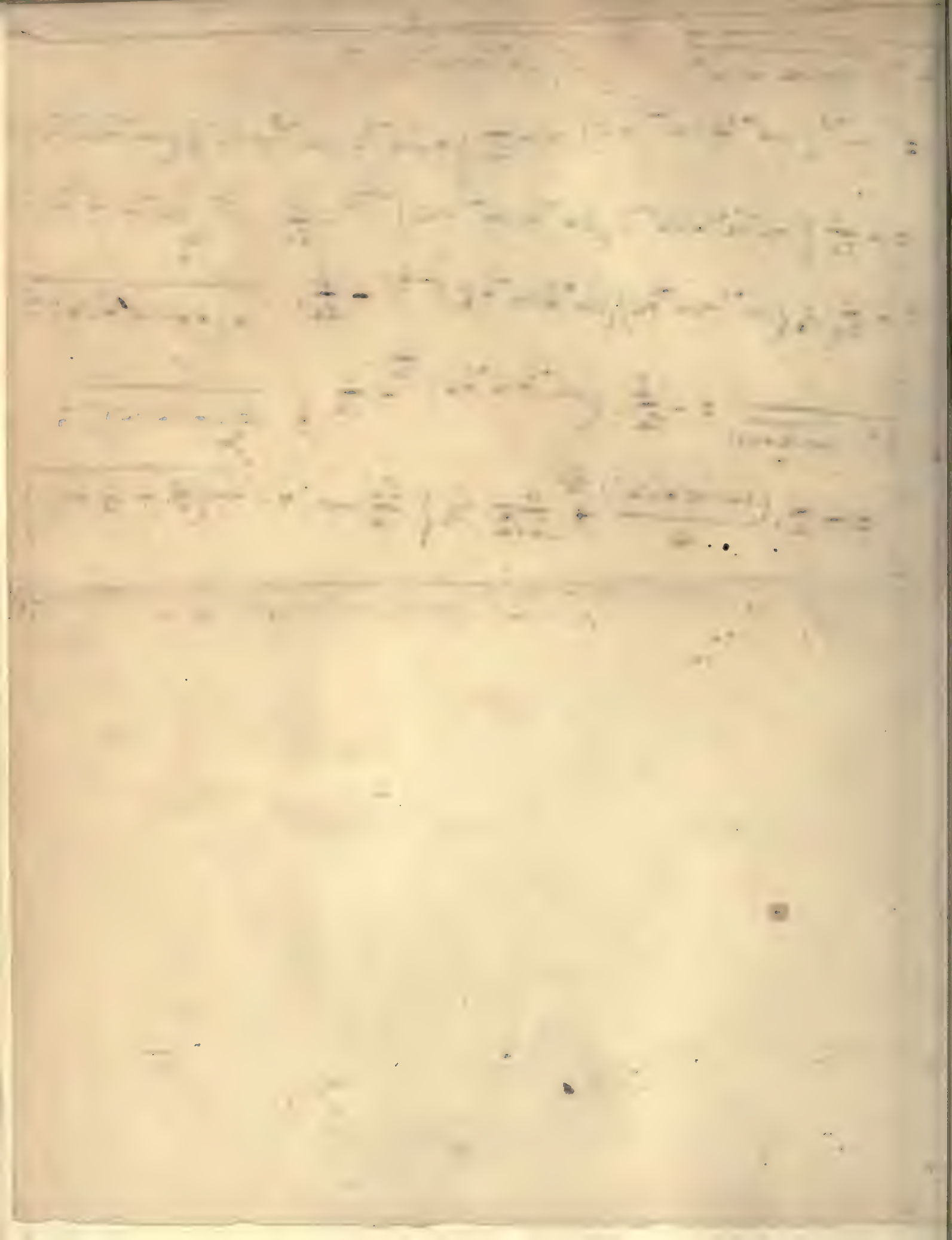
$$= x^{-3}\left(ax^{-2}+bx^{-1}+c\right) = -\frac{1}{2a}\left\{-2ax^{-3}-bx^{-2}+bx^{-2}\right\}\left(ax^{-2}+bx^{-1}+c\right)^{-\frac{1}{2}}$$

$$= -\frac{1}{2a}\left\{-2ax^{-3}-bx^{-2}\right)\left(ax^{-2}+bx^{-1}+c\right)^{-\frac{1}{2}} - \frac{b}{2a}\cdot x^{-2}\left(ax^{-2}+bx^{-1}+c\right)^{-\frac{1}{2}}.$$

$$= -\frac{1}{2a}\left(2x\left(ax^{-2}+bx^{-1}+c\right)\left(ax^{-2}+bx^{-1}+c\right)^{-\frac{1}{2}} - \frac{b}{2a}\cdot\frac{1}{x\left(a+bx^{1}+cx^{2}\right)^{\frac{1}{2}}}\right.$$

$$\int_x \frac{1}{x^2\sqrt{a+bx+cx^2}} = -\frac{2}{2a}\cdot\left(ax^{-2}+bx^{-1}+c\right)^{\frac{1}{2}} - \frac{b}{2a}\int_x \frac{1}{x\left(a+bx+cx^2\right)^{\frac{1}{2}}}.$$

$$= -\frac{1}{a}\cdot\frac{\left(a+bx+cx^2\right)^{\frac{1}{2}}}{x} + \frac{b}{2a\sqrt{a}}\log_x\left\{\frac{2a}{x}+b+\sqrt{4a\cdot\left(\frac{a}{x^2}+\frac{b}{x}+c\right)}\right\}.$$



$$u = \frac{x^2+ac}{x\sqrt{(a^2-x^2)(x^2-c^2)}} = \frac{x}{\sqrt{(a^2-x^2)(x^2-c^2)}} + \frac{ac}{x\sqrt{(a^2-x^2)(x^2-c^2)}}$$

$$\frac{x}{\sqrt{(a^2-x^2)(x^2-c^2)}} = \int x \frac{dx\sqrt{x^2-c^2}}{\sqrt{a^2-c^2-(x^2-c^2)}} = hu^{-1}\int \frac{z^2-c^2}{a^2-c^2}$$

$$c\int_x \frac{1}{x\sqrt{(a^2-x^2)(a^2-c^2)}}$$

Let $x = \frac{1}{\sqrt{z}} \quad \therefore \frac{1}{x} = -\frac{1}{2}\frac{dz}{z}$

$$\therefore ac\int_x \frac{1}{x\sqrt{(a^2-x^2)(x^2-c^2)}} = -\frac{ac}{2}\int_z \frac{1}{z\sqrt{(a^2-\frac{1}{z})(\frac{1}{z}-c^2)}}$$

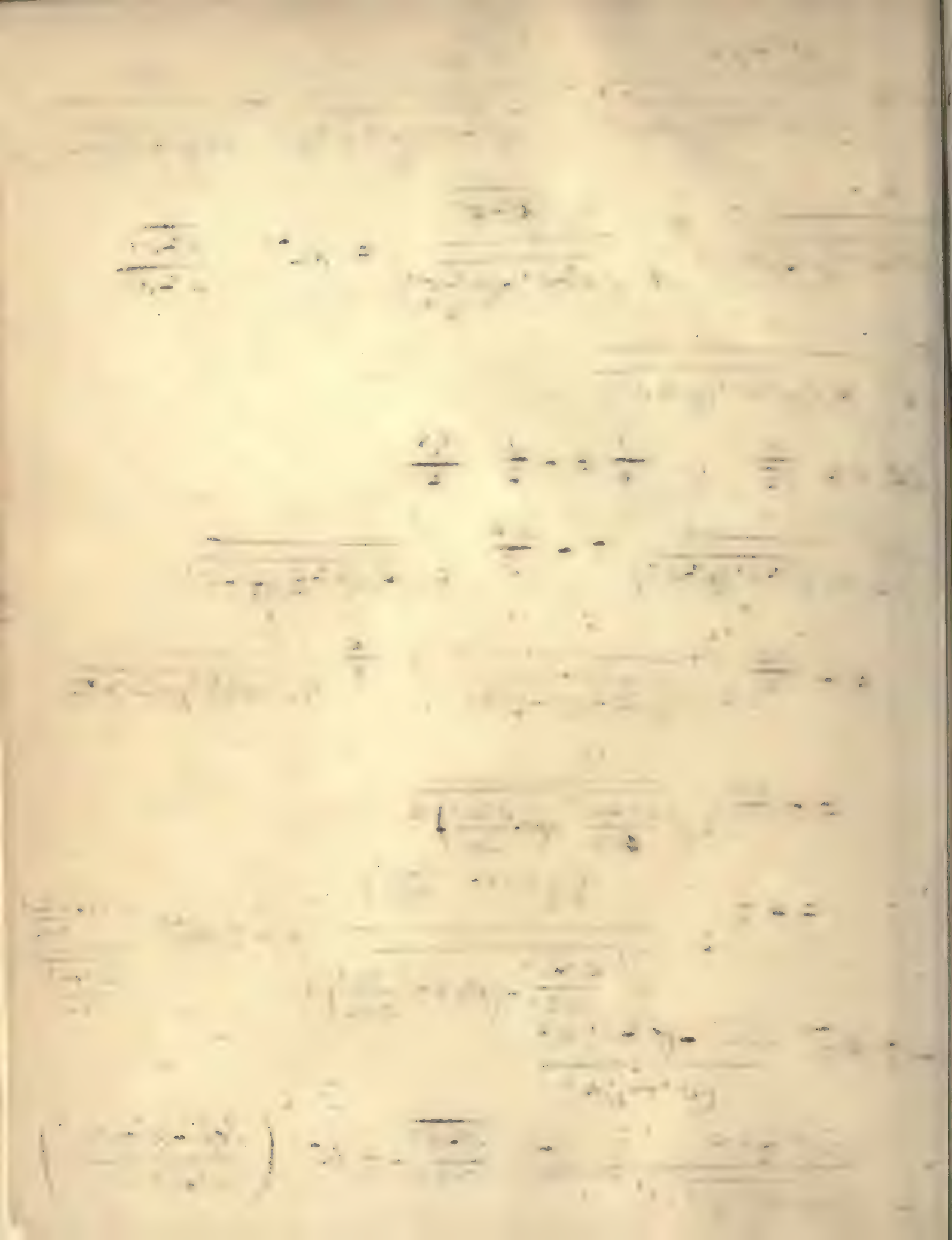
$$= -\frac{ac}{2}\int_z \frac{1}{\sqrt{(a^2z-1)(1-c^2z)}} = \frac{ac}{2}\int_z \frac{1}{\sqrt{-c^2a^2z^2+(c^2+a^2)z-1}}$$

$$= -\frac{ac}{2}\int_z \frac{1}{\sqrt{\frac{c^4+a^4}{4c^2a^2}-\left(caz-\frac{c^2+a^2}{2ca}\right)^2}}$$

$$= -\frac{1}{2}\int_z \frac{d_z\left(caz-\frac{c^2+a^2}{2ca}\right)}{\sqrt{\frac{c^4+a^4}{4c^2a^2}-\left(caz-\frac{c^2+a^2}{2ca}\right)^2}} = -\frac{1}{2}hu^{-1}\frac{caz-\frac{c^2+a^2}{2ca}}{\sqrt{\frac{c^4+a^4}{2ca}}}$$

$$= -\frac{1}{2}hu^{-1}\frac{2ca^2-(c^2+a^2)x^2}{(c^4+a^4)x^2}$$

$$\therefore \int_x \frac{x^2+ac}{x\sqrt{(a^2-x^2)(x^2-c^2)}} = hu^{-1}\sqrt{\frac{x^2-c^2}{a^2-c^2}} - \frac{1}{2}hu^{-1}\left(\frac{2ca^2-(c^2+a^2)x^2}{(c^4+a^4)x^2}\right)$$



$$\int \frac{(\alpha+\beta x)\sqrt{a+bx^2}}{c+ex^2}\,dx = \int \frac{(\alpha+\beta x)(a+bx^2)}{(c+ex^2)\sqrt{a+bx^2}}\,dx$$

$$= \int \frac{\frac{ba}{e}}{\sqrt{a+bx^2}}\,dx + \int \frac{\frac{\beta b}{e}}{\sqrt{a+bx^2}}\,dx + \left(\frac{\alpha a e - b\beta c}{e}\right)\int \frac{dx}{(c+ex^2)\sqrt{a+bx^2}}$$

$$+ \frac{\beta}{e}(\alpha e - bc)\int \frac{dx}{(c+ex^2)\sqrt{a+bx^2}}$$

$$= \frac{\beta}{e}\sqrt{a+bx^2} + \frac{\beta\sqrt{b}}{e}\log_e\left(x+\sqrt{\frac{a}{b}+x^2}\right)$$

$$+ \frac{\alpha a e - b\beta c}{e}\int \frac{dz}{bc+e(z^2-a)} \qquad \text{where } z = \sqrt{a+bx^2}$$

$$+ \frac{\beta}{e}\cdot\frac{(\alpha e - \beta c)}{\sqrt{c(\alpha e - c\beta)}}\tan^{-1}\sqrt{\frac{c(a+bx^2)}{a^2(\alpha e - c\beta)}}$$

$$u = \frac{1}{(h+gx)\sqrt{a+bx+cx^2}}$$

Let $h+gx = z \quad \therefore \frac{g}{h+gx} = \frac{dz}{z}$ or $\frac{1}{h+gx} = \frac{1}{g}\cdot\frac{dz}{z}$

$$\therefore \int_u \frac{1}{(h+gx)\sqrt{a+bx+cx^2}} = \frac{1}{g}\int_z \frac{1}{z\sqrt{a+b\left(\frac{z-h}{g}\right)+c\left(\frac{z-h}{g}\right)^2}}$$

$$= \int_z \frac{1}{z\sqrt{cz^2+(bg-2hc)z+h^2c+ag^2-bhg}}$$

$$= \frac{1}{\sqrt{h^2c+ag^2-bhg}}\log_z \left\{ \frac{2(h^2c+ag^2-bhg)}{h+gx} + (bg-2hc) + \right.$$

$$\left. + \sqrt{4(h^2c+ag^2-bhg)\left\{ \frac{h^2c+ag^2-bhg}{(h+gx)^2} + \frac{bg-2hc}{h+gx} + c \right.} \right.$$

$$u = \int_u (b-x)^{\frac{m}{n}}\cdot(a-x) = \int_u \{(a-b)+(b-x)\}(b-x)^{\frac{m}{n}}$$

$$= \int_u (a-b)(b-x)^{\frac{m}{n}} + \int_u (b-x)^{\frac{m+n}{n}}$$

$$= \frac{n}{m+n}(b-a)(b-x)^{\frac{m+n}{n}} - \frac{n}{m+2n}(b-x)^{\frac{m+2n}{n}}$$

$$\int (x^2+a)\sqrt{x^2+b}\,dx = \int (x^2+b+a-b)\sqrt{x^2+b}\,dx$$

$$= \int (x^2+b)^{\frac{3}{2}}\,dx + (a-b)\int (x^2+b)^{\frac{1}{2}}\,dx.$$

$$\int (x^2+b)^{\frac{3}{2}}\,dx = x(x^2+b)^{\frac{3}{2}} - 3\int (x^2+b-b)(x^2+b)^{\frac{1}{2}}\,dx$$

$$= x(x^2+b)^{\frac{3}{2}} - 3\int (x^2+b)^{\frac{3}{2}}\,dx - 3b\int (x^2+b)^{\frac{1}{2}}\,dx.$$

$$\therefore \int (x^2+b)^{\frac{3}{2}}\,dx = \frac{x}{4}(x^2+b)^{\frac{3}{2}} - \frac{3b}{4}\int (x^2+b)^{\frac{1}{2}}\,dx.$$

$$\text{Let } \int (x^2+b)^{\frac{1}{2}}\,dx = x(x^2+b)^{\frac{1}{2}} - \int \frac{x^2+b-b}{(x^2+b)^{\frac{1}{2}}}\,dx.$$

$$= \frac{x}{2}(x^2+b)^{\frac{1}{2}} + \frac{b}{2}\log_e\left(x+\sqrt{x^2+b}\right)$$

$$\therefore \left(a-\tfrac{1}{4}b\right)\int (x^2+b)^{\frac{1}{2}}\,dx = \left(a-\tfrac{1}{4}b\right)\left\{\frac{x}{2}(x^2+b)^{\frac{1}{2}} + \frac{b}{2}\log_e\left(x+\sqrt{x^2+b}\right)\right.$$

$$\therefore \int (x^2+a)\sqrt{x^2+b}\,dx = \frac{x}{4}(x^2+b)^{\frac{3}{2}} + \left(a-\tfrac{1}{4}b\right)\left\{\frac{x}{2}(x^2+b)^{\frac{1}{2}} + \frac{b}{2}\log_e\left(x+\sqrt{x^2+b}\right)\right.$$

$$\frac{1}{(x+a)\sqrt{x-x^2}} = \int \frac{x^{-2}}{1+ax^{-1}\sqrt{x^{-1}-1}}$$

$$\text{Let } \sqrt{x^{-1}-1} = z \therefore \frac{dx}{x} = -\frac{1}{2}\frac{x^{-2}}{\sqrt{x^{-1}-1}}$$

$$\therefore \int \frac{x^{-2}}{(1+ax^{-1})\sqrt{x^{-1}-1}} = -2\int \frac{1}{1+a(z^2+1)} = -\frac{2}{a}\int \frac{1}{\frac{1+a}{a}+z^2}$$

$$= -\frac{2}{\sqrt{a^2+a}} \tan^{-1} \frac{z\sqrt{a}}{\sqrt{1+a}}$$

$$= -\frac{2}{\sqrt{a^2+a}} \tan^{-1} \frac{\sqrt{a-ax}}{\sqrt{ax+x}}.$$

$$\text{Let } l = \frac{\ln z - \ln x}{z - x}$$

$$\therefore \ln z = \ln x + \left(\frac{\ln z - \ln x}{z - x}\right)(z - x) \qquad (*)$$

$$\frac{d}{dx}\ln z = 0 = \cos x + \frac{(z-x) - \cos x + \ln z - \ln x}{(z-x)^2}(z-x) - \frac{\ln z - \ln x}{z-x} \qquad (1)$$

$$\frac{d^2}{dx^2}\ln z = 0 = -\ln x + \frac{\{z-x\}^2\{(z-x)\ln x + \cos x - \cos x\} + 2(z-x)\{(z-x) - \cos x + \ln z - \ln x\}}{(z-x)^4}(z-x) - 2\left\{\frac{(z-x) - \cos x + \ln z - \ln x}{(z-x)^2}\right\}$$

$$= -\ln x + \frac{(z-x)^3 \ln x + 2(z-x)\{(z-x) - \cos x + \ln z - \ln x\}}{(z-x)^4}(z-x) - 2\left\{\frac{(z-x) - \cos x + \ln z - \ln x}{(z-x)^2}\right\} \qquad (2)$$

$$\frac{d^3}{dx^3}\ln z = 0 = -\cos x + \frac{(z-x)^4\{(z-x)^3\cos x + 3(z-x)^2 \ln x\} + (z-x)^4 \cdot 2(z-x)\{(z-x)\ln x\} - (z-x)^4 2\{(z-x) - \cos x + \ln z - \ln x\}}{(z-x)^7}$$

$$+ \frac{4(z-x)^3\{(z-x)^3 \ln x + 2(z-x)\{(z-x) - \cos x + \ln z - \ln x\}\}}{(z-x)^7}$$

$$- 3\left\{\frac{(z-x)^3 \ln x + 2(z-x)\{(z-x)\cos x + \ln z - \ln x\}}{(z-x)^4}\right\}$$

$$= -\cos x + \frac{(z-x)^7\cos x - 3(z-x)^6 \ln x + 2(z-x)^6 \ln x + 2(z-x)^5\cos x - 2(z-x)^4\{\ln z - \ln x\}}{(z-x)^7}$$

$$+ \frac{4(z-x)^6 \ln x + 8(z-x)^5 \cdot \cos x + 4(z-x)^4 \ln z - \ln x}{(z-x)^7}$$

$$\frac{\qquad\qquad}{(z-x)^4} = A$$

$$0 = -\cos x + \frac{(z-x)^5\cos x + 3(z-x)^6\sin x + 6(z-x)^5\cos x + 6(z-x)^4(\sin z - \sin x)}{(z-x)^6}(z-x)^4 \quad -(A)$$

$$0 = -\cos x + \frac{(z-x)^3\cos x + 3(z-x)^2\sin x + 6(z-x)\cos x + 6(\sin z - \sin x)}{(z-x)^4}(z-x) - \frac{3(z-x)^2\sin x + 6(z-x)\cos x + 6(\sin z - \sin x)}{(z-x)^3}\frac{(z-x)^2}{2} + \frac{(z-x)^3}{1\cdot 2\cdot 3}(3)$$

Multiply (1), (2), (3) by $\frac{z-x}{1}, \left(\frac{z-x}{2}\right)^2$ & — add we have

$$0 = (z-x)\cos x + \frac{(z-x)\cos x + \sin z - \sin x}{(z-x)^2}(z-x)^2 - \frac{\sin z - \sin x}{(z-x)}(z-x) \quad = (B)$$

$$0 = -\frac{(z-x)^2}{1\cdot 2}\sin x + \frac{(z-x)^3\sin x + 2(z-x)\{(z-x)\cos x + \sin z - \sin x\}}{(z-x)^4}\frac{(z-x)^3}{1\cdot 2} - \frac{(z-x)\cos x + \sin z - \sin x}{(z-x)^2}\frac{(z-x)^2}{2}$$

$$0 = -\frac{(z-x)^3}{1\cdot 2\cdot 3}\cos x + \frac{(z-x)^3\cos x + 3(z-x)^2\sin x - 6(z-x)\cos x + 6(\sin z - \sin x)}{(z-x)^4}\frac{(z-x)^4}{1\cdot 2\cdot 3} \quad (B)$$

$$-\frac{3\{(z-x)^2\sin x + 2(z-x)\{(z-x)\cos x + \sin z - \sin x\}\}}{1\cdot 2\cdot 3}\frac{(z-x)^3}{1\cdot 2\cdot 3} \quad \& \quad — \quad (\delta)$$

By adding together $\alpha, \beta, \gamma, \delta$ we have.

$$\sin z = \sin x + (z-x)\cos x - \frac{(z-x)^2}{1\cdot 2}\sin x - \frac{(z-x)^3}{1\cdot 2\cdot 3}\cos x + (B)$$

$$\therefore \sin(x+h) = \sin x + \frac{h}{1}\cos x - \frac{h^2}{1.2.}\sin x - \frac{h^3}{1.2.3}\cos x + \frac{d^3}{x^3}P.\frac{h^4}{1.2.3.}\quad x < \frac{}{1.2.3.}$$

Also from (3) we have

$$0 = -\cos x + \frac{(z-x)^3\cos x + 3(z-x)^2\sin x + 6(z-x) - \cos x + 6(\sin z - \sin x)}{(z-x)^4}(z-x) - \frac{3\{(z-x)^2\sin x + 6(z-x) - \cos x + 6(\sin z - \sin x)\}}{(z-x)^3}$$

$$0 = -\cos x(z-x)^2 + \frac{(z-x)^3\cos x + 3(z-x)^2\sin x + 6(z-x) - \cos x + 6(\sin z - \sin x)}{(z-x)^4}(z-x)^3 - \frac{3(z-x)^2\sin x + 6(z-x) - \cos x + 6(\sin z - \sin x)}{(z-x)^3}(z-x)$$

$$= -\cos x(z-x)^2 + \frac{1}{2}\left\{\frac{(z-x)^2\sin x + 2\{(z-x) - \cos x + \sin z - \sin x\}}{(z-x)^3}(z-x)^3\right\}$$

Here $-\cos x = d^n f(x)$

$\therefore$ as x changes from $-\pi$ to $\frac{2\pi}{2}\cdot\frac{3\pi}{2}$ $d^n f(x)$ becomes the greatest and least values $+1, 0, -1$

$\therefore A = 1$ and $B = -1$ between the limits π and $\frac{3\pi}{2}$. where $\frac{3\pi}{2} = z$

$$\therefore A(z-x)^2 + \frac{1}{2}\left\{\frac{(z-x)^2\sin x + 2\{z-x) - \cos x + \sin z - \sin x\}}{(z-x)^3}(z-x)^3\right\}$$

$$= (z-x)^2 + \left\{\frac{(z-x)^3\cos\pi + 6(z-x) - \cos\pi\} - 6(z-x) - \cos\pi\}}{(z-x)}\right\} = (z-x)^2\cdot\cos\pi + (z-x)^2$$

$$= (z-x)^2(-1) + (z-x)^2 = 0$$

$\therefore A(z-x)^2 + \frac{1}{2}\left\{\frac{(z-x^2)\sin x + 2\{(z-x) - \cos x + \sin z - \sin x\}}{(z-x)^3}(z-x)^3\right\}$ is ~~always neg. when the~~ which is never positive ~~proposed limits~~

and $B(z-x)^2 + x$ — is always positive between the same limits

$= 0 + 0 = 0$ which is never negative.

Assume $F(z) = -\frac{A}{3}(z-x)^3 + \frac{(z-x)^2 \sin x + 2\{(z-x) - \cos x + \sin z - \sin x\}}{(z-x)^3}(z-x)^3$

$\therefore F(z) = 0$ for changing x into z $-\frac{A}{3}(z-x)^3 x -$ all vanished

$\therefore \frac{F(z) - F(x)}{z-x} = \frac{A}{3}(z-x)^2 \frac{(z-x)^2 \sin x + 2\{(z-x) - \cos x + \sin z - \sin x\}}{(z-x)^3}(z-x)^2$

also $d_x F(x) = A.(z-x)^2 + d_x\left\{\frac{(z-x)^2 \sin x + 2\{(z-x) - \cos x + \sin z - \sin x\}}{(z-x)^3}(z-x)^3\right\}$

Now assigns x lies between $-\frac{3\pi}{2}$ and π the sign of $d_x Fx$ will not alter.

$\therefore \frac{F\left(\frac{3\pi}{2}\right) - F(\pi)}{\frac{3\pi}{2} - \pi}$ and $d_x F(x)$ have the same sign.

$\therefore \frac{\pi^n}{2^n} \frac{\left(\frac{\pi}{2}\right)^2}{3} - \left\{\frac{\pi}{2}^2 \sin \pi + 2\left\{\frac{\pi}{2}\right) - \cos \pi + \sin \frac{3\pi}{2} - \sin \pi\right\} = \frac{\pi^2}{4.3} - 2\left(\frac{\pi}{2} - 1\right)$ which is negat.

also. $\left(\frac{\pi}{2}\right)^n + \left\{(z-\alpha)^4(-1)\right\}$ is negative

$\therefore \frac{A.(z-x_1)^n}{n+1} \cdot d_{x=x_1}^n P(z-x_1)^n$ and $A(z-x)^n + d_x\left\{d_x^n P(z-x)^{n+1}\right\}$ are both negat.

and $\frac{1}{3}$, $\frac{0}{3}$ are the limits of the value of $d^2 P$.

$\therefore$ the rem. after the first 4 terms of the expansion of $\sin(\pi + h)$ lies between $\frac{hh}{3}$, and zero.

$f(x+h) = f(x) + d_x f(x)\frac{h}{1} + d_x^2 f(x)\frac{h^2}{2} + d_x^3 f(x)\frac{h^3}{3} \cdots$

$f(x, y+k) = f(x,y) + d_y f(x,y)\frac{k}{1} + d_y^2 f(x,y)\frac{k^2}{2} + d_y^3 f(x,y)\frac{k^3}{3}$

write $x+h$ for x

$f\{(x+h)+y+k\} = f(x+h, y) + d_y f(x+h, y)\frac{k}{1} + d_y^2 f(x+h, y)\frac{k^2}{2} + d_y^3 f(x+h, y)\frac{k^3}{3}$

But

$f(x+h, y) = f(x,y) + d_x f(x,y)\frac{h}{1} + d_x^2 f(x,y)\frac{h^2}{2} + d_x^3 f(x,y)\frac{h^3}{3} \cdots$

$d_y f(x+h, y) = d_y f(x,y) + d_x d_y f(x,y)\frac{h}{1} + d_x^2 d_y f(x,y)\frac{h^2}{2} + d_x^3 d_y f(x,y)\frac{h^3}{3}$

$d_y^2 f(x+h, y) = d_y^2 f(x,y) + d_x d_y^2 f(x,y)\frac{h}{1} + d_x^2 d_y^2 f(x,y)\frac{h^2}{2}$

$d_y^3 f(x+h, y) = d_y^3 f(x,y) + d_x d_y^3 f(x,y)\frac{h}{1}$

$f(x+h, y+k) = f(x,y) + d_x f(x,y)\frac{h}{1} + d_x^2 f(x,y)\frac{h^2}{2} + d_x^3 f(x,y)\frac{h^3}{3} \cdots d_x^n f(x,y)\frac{h^n}{n}$

$+ d_y f(x,y)\frac{k}{1} + d_x d_y f(x,y)\frac{hk}{11} + d_x^2 d_y f(x,y)\frac{h^2}{2}\frac{k}{1} \cdots + d_x^n d_y f(x,y)\frac{h^n}{n}$

$+ d_y^2 f(x,y)\frac{k^2}{12} + d_x d_y^2 f(x,y)\frac{h}{1}\frac{k^2}{2} \cdots$

Similarly

$f(x, y, z+l) = f(x,y,z) + d_z f(x,y,z)\frac{z}{1} + d_z^2 f(x,y,z)\frac{z^2}{2} + d_z^3 f(x,y,z)\frac{z^3}{3}$

write $y+k$ for y

$f(x, y+k, z+l) = f(x, y+k, z) + d_z f(x, y+k, z)\frac{z}{1} + d_z^2 f(x, y+k, z)\frac{z^2}{2} + \cdots$

But $f(x, y+k, z) = f(x,y,z) + d_y f(x,y,z)\frac{k}{1} + d_y^2 f(x,y,z)\frac{k^2}{2} + d_y^3 f(x,y,z)\frac{k^3}{3}$

$d_z f(x, y+k, z) = d_z f(x,y,z) + d_y d_z f(x,y,z)\frac{k}{1} + d_y^2 d_z f(x,y,z)\frac{k^2}{2} \cdots$

$d_z^2 f(x, y+k, z) = d_z^2 f(x,y,z) + d_y d_z^2 f(x,y,z)\frac{k}{1} \cdots$

$f(x, y+k, z+l) = f(x,y,z) + d_y f(x,y,z)\frac{k}{1} + d_y^2 f(x,y,z)\frac{k^2}{2} + d_y^3 f(x,y,z)\frac{k^3}{3}$

$+ d_z f(x,y,z)\frac{z}{1} + d_y d_z f(x,y,z)\frac{kz}{11} + d_y^2 d_z f(x,y,z)\frac{k^2}{2}\frac{z}{1}$

$+ d_z^2 f(x,y,z)\frac{z^2}{12} + d_y d_z^2 f(x,y,z)\frac{k}{1}\frac{z^2}{12} \cdots$

Change x into $x+h$ and we have

$$f(x+h, y+k, z+l) = f(x+h, y, z) + d_y f(x+h, y, z)\frac{k}{1} + d_y^2 f(x+h, y, z)\frac{k^2}{2} + \cdots$$
$$d_x f(x+h, y, z)\frac{x}{1} + d_y d_x f(x+h, y, z)\frac{k}{1}\frac{x}{1} + d_y^2 d_x f(x+h, y, z)\frac{k^2}{2}\frac{x}{1}$$
$$+ d_x^2 f(x+h, y, z)\frac{x^2}{1\cdot 2} + d_y d_x^2 f(x+h, y, z)\frac{k}{1}\frac{x^2}{1\cdot 2} \cdots$$

$$f(x+h, y, z) = f(x, y, z) + d_x f(x, y, z)\frac{h}{1} + d_x^2 f(x, y, z)\frac{h^2}{2} + d_x^3 f(x, y, z)\frac{h^3}{3} \cdots$$

$$d_y f(x+h, y, z) = d_y f(x, y, z) + d_x d_y f(x, y, z)\frac{h}{1} + d_x^2 d_y f(x, y, z)\frac{h^2}{2}$$

$$d_y^2 f(x+h, y, z) = d_y^2 f(x, y, z) + d_x d_y^2 f(x, y, z)\frac{h}{1}$$

$$d_z f(x+h, y, z) = d_z f(x, y, z) + d_x d_z f(x, y, z)\frac{h}{1} + d_x^2 d_z f(x, y, z)\frac{h^2}{2}$$

$$d_y d_z f(x+h, y, z) = d_y d_z f(x, y, z) + d_x d_y d_z f(x, y, z)\frac{h}{1}$$

$$d_y^2 d_z f(x+h, y, z) = d_y^2 d_z f(x, y, z) + d_x d_y^2 d_z f(x, y, z)\frac{h}{1}$$

$$d_z^2 f(x+h, y, z) = d_z^2 f(x, y, z) + d_x d_z^2 f(x, y, z)\frac{h}{1} + d_x^2 d_z^2 f(x, y, z)\frac{h^2}{2}$$

$$d_y d_z^2 f(x+h, y, z) = d_y d_z^2 f(x, y, z) + d_x d_y d_z^2 f(x, y, z)\frac{h}{1} + d_x^2 d_y d_z^2 f(x, y, z)\frac{h^2}{2}$$

$$f(x+h, y+k, z+l) = f(x, y, z) + d_x f(x, y, z)\frac{h}{1} + d_x^2 f(x, y, z)\frac{h^2}{2} + d_x^3 f(x, y, z)\frac{h^3}{3}$$
$$+ d_y f(x, y, z)\frac{k}{1} + d_x d_y f(x, y, z)\frac{hk}{1} + d_x^2 d_y f(x, y, z)\frac{hk^2}{2}$$
$$+ d_y^2 f(x, y, z)\frac{k^2}{2} + d_x d_y^2 f(x, y, z)\frac{k^2 h}{1\cdot 2} + d_x^2 d_y^2 f(x, y, z)\frac{h^2}{1\cdot 2}\frac{k^2}{2}$$
$$+ d_z f(x, y, z)\frac{z}{1} + d_x d_z f(x, y, z)\frac{hz}{1} + d_x^2 d_z f(x, y, z)\frac{h^2}{2}\frac{z}{1}$$
$$+ d_y d_z f(x, y, z)\frac{kz}{1} + d_x d_y d_z f(x, y, z)\frac{h}{1}\cdot\frac{kz}{1}$$
$$+ d_y^2 d_z f(x, y, z)\frac{k^2}{2}\cdot\frac{z}{1}$$
$$+ d_z^2 f(x, y, z)\frac{z^2}{1\cdot 2} + d_x d_z^2 f(x, y, z)\frac{hz^2}{1\cdot 2} + d_x^2 d_z^2 f(x, y, z)\frac{h^2}{2}\frac{z^2}{1\cdot 2}$$
$$+ d_y d_z^2 f(x, y, z)\frac{kz^2}{1\cdot 2} + d_x d_y d_z^2 f(x, y, z)\frac{h}{1}\cdot\frac{kz^2}{1\cdot 2}$$

Expand $f(x+h, y+k, z+l)$ to terms involving $h^2, hk, \cdots$ $+ d_y^2 d_z^2 f(x, y, z)\frac{k^2}{2}\frac{z^2}{1\cdot 2}$

$P = \frac{f(z) - fx}{z-x} \quad \therefore f(z) = f(x) + P.(z-x)\,; \quad 0 = d_x f(x) + d_x P(z-x) - P. \quad (1)$

$$0 = d_x^2 f(x) + d_x^2 P(z-x) - 2d_x P. \quad (2)$$

(1) $0 = d_x f(x) + d_x \{f(z) - f(x)\} - \frac{f(z) - f(x)}{z-x}$ $\qquad 0 = d_x^3 f(x) + d_x^3 P(z-x) - 3 d_x P \quad (3)$

(2) $0 = d_x^2 f(x) + d_x^2 \{f(z) - f(x)\} - 2 d_x \frac{f(z) - f(x)}{z-x}$ $\qquad$ It is evident that (1) (2) (3) etc may be written as on the right or left hand d.

(3) $0 = d_x^3 f(x) + d_x^3 \{f(z) - f(x)\} - 3 d_x \frac{f(z) - f(x)}{z-x}$

Multiply (1)(2)(3) &c by $\frac{z-x}{1}, \frac{(z-x)^2}{\lfloor 2}, \frac{(z-x)^3}{1.2.3}$ &c

$0 = (z-x) d_x f(x) + d_x (z-x) . f(x) - (z-x) d_x f(x) - f(x) + f(x)$

or $f(z) = f(x) + d_x (z-x) f(x) \quad (A)$

$$0 = \frac{(z-x)^2}{\lfloor 2} d_x^2 f(x) + \frac{(z-x)^2}{2} d_x^2 f(z) - \frac{(z-x)^2}{2} d_x^2 f(x) - (z-x) d_x f(z) + (z-x) d_x f(x)$$

$$0 = \frac{z^2 - 2zx + x^2}{2} d_x^2 f(x) - (z-x) d_x f(x) + (z-x) d_x f(x) \quad B$$

and $f(z) = f(x) + d_x (z-x) f(x)$

by addition of A and B we have

$$f(z) = f(x) + \frac{z^2 - 2zx + x^2}{2} d_x^2 f(x) + (z-x) d_x f(x) \quad (C)$$

$$0 = \frac{(z^2 - 2zx + x^2)(z-x)}{1.2.3} d_x^3 f(x) - \frac{3}{6}(z-x)^2 d_x^2 f(x) + \frac{3}{6}(z-x)^2 d_x^2 f(x) \quad D$$

$$0 = \frac{(z-x)^4}{\lfloor 4} d_x^4 f(x) - \frac{4}{\lfloor 4} (z-x)^3 d_x^3 f(x) + \frac{4}{\lfloor 4} (z-x)^3 d_x^3 f(x) \quad E$$

$$0 = \frac{(z-x)^5}{\lfloor 5} d_x^5 f(x) - \frac{5}{\lfloor 5} (z-x)^4 d_x^4 f(x) + \frac{5}{\lfloor 5} (z-x)^4 d_x^4 f(x) \quad F$$

&c &c &c

$$0 = \frac{(z-x)^{n-1}}{\lfloor n-1} d_x^{n-1} f(x) - \frac{n-1}{\lfloor n-1} (z-x)^{n-2} d_x^{n-2} f(z) + \frac{n-1}{\lfloor n-1} (z-x)^{n-2} d_x^{n-2} f(x)$$

$$0 = \frac{(z-x)^n}{\lfloor n} d_x^n f(x) - \frac{n}{\lfloor n} (z-x)^{n-1} d_x^{n-1} f(x) + \frac{n}{\lfloor n} (z-x)^{n-1} d_x^{n-1} f(x)$$

By addition we have

$$f(z) = f(x) + d_x f(x) \frac{z-x}{1} + d_x^2 f(x) \frac{(z-x)^2}{2} + d_x^3 f(x) \frac{(z-x)^3}{\lfloor 3} + d_x^4 f(x) \frac{(z-x)^4}{\lfloor 4} + \&c$$

$$+ d_x^n f(x) \frac{(z-x)^n}{\lfloor n} + d_x^n P. \frac{(z-x)^{n+1}}{\lfloor n}$$

If $z - x = h$

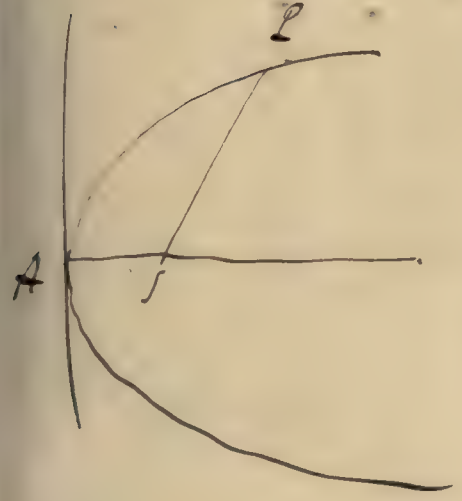
$$\therefore f(x+h) = f(x) + d_x f(x) \frac{h}{1} + d_x^2 f(x) \frac{h^2}{\lfloor 2} + d_x^3 f(x) \frac{h^3}{\lfloor 3} + \&c \quad \ldots \quad d_x^n f(x) \frac{h^n}{\lfloor n}$$

$$+ d_x^n P. \frac{h^{n+1}}{\lfloor n}$$

Ellipse $y^2 = 4ax$.

$$\frac{dy}{dx} = \sqrt{\frac{a}{x}} \cdot \frac{dy}{dx} = -\frac{1}{2}\frac{a^{\frac{1}{2}}}{x^{\frac{3}{2}}}.$$

$$r = -\frac{1}{a^2 y}\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}} = \frac{2x^{\frac{3}{2}}}{a^{\frac{1}{2}}}\left\{\frac{a+x}{x}\right\}^{\frac{3}{2}}.$$



$$= \frac{2(a+x)^{\frac{3}{2}}}{a^{\frac{1}{2}}}$$

$$= \frac{2(SP)^{\frac{3}{2}}}{AS^{\frac{1}{2}}}.$$

$u = x^3 y^2 (6 - x - y)$.

$\frac{du}{dx} = 3x^2y^2(6-x-y) - x^3y^2 = x^2y^2(18 - 4x - 3y) = 0$ if $x = 0$ (

$\frac{du}{dy} = 2x^3y(6-x-y) - x^3y^2 = x^3y(12 - 2x - 3y) = 0$ if $x = 0$.

$\frac{d^2u}{dx^2} = 6xy^2(6-x-y) - 3x^2y^2 - 3x^2y^2$

$= xy^2(36 - 12x - 6y) = 0$ if $x = 0$.

$\frac{d^2u}{dy^2} = 2x^3(6-x-y) - 2x^3y - 6x^3y = 0$ if $x = 0$.

Taking $x = 0$, $y = 0$. $\frac{d^2u}{dy^2} \cdot \frac{d^2u}{dx^2}$ is not greater than $\left(\frac{d^2u}{dx\,dy}\right)^2$

$\therefore x = 0$, $y = 0$ give neither max nor min.

Also for (1) $18 - 4x - 3y = 0$.

$$x = \frac{18 - 3y}{4} = \left(\frac{12 - 3y}{2}\right)$$

$\therefore 36 - 6y = 48 - 12y$.

$6y = 12 \quad \therefore y = 2 \quad \therefore x = 3$.

$\frac{d^2u}{dx^2} = \frac{d^2u}{dy^2}$ are both neg. and $\frac{d^2u}{dx^2} \cdot \frac{d^2u}{dy^2} > \frac{d^2u}{dx\,dy}$.

Therefore u is a max. if $y = 2$, $x = 3$

$x^5 - 3ay^4 - x^2y^3 = 0$. A

$10x^4 - 12ay^3 \frac{dy}{dx} - 2xy^3 - 3x^2y^2 \frac{dy}{dx} = 0$. (1)

$$\therefore \frac{dy}{dx} = -\frac{2xy^3 - 10x^4}{12ay^3 + 3x^2y^2}$$

$$= \frac{10x^4 - 2xy^3}{12ay^3 + 3x^2y^2}$$

$\therefore \frac{dy}{dx} = 0$ when $10x^4 = 2xy^3$ $x = 0$ or $x^3 = \left(\frac{1}{5}\right)y^3$

Diff (1) with resp to x. $x = y\sqrt[3]{\frac{1}{5}}$.

$$40x^3 - 12ay^3 \frac{d^2y}{dx^2} + 36ay^2\left(\frac{dy}{dx}\right)^2 - 2y^3 - 6xy^2\frac{dy}{dx}$$

$$- 3x^2y^2 \cdot \frac{d^2y}{dx^2} - 6xy^2\frac{dy}{dx} - 6x^2\left(\frac{dy}{dx}\right)^2 = 0$$

Putting x max or min $\frac{dy}{dx} = 0$.

$\therefore 40x^3 - 12ay^3\frac{d^2y}{dx^2} - 2y^3 - 3x^2y^2\frac{d^2y}{dx^2} = 0$

$$\therefore \frac{d^2y}{dx^2} = -\frac{2y^3 - 40x^3}{12ay^3 - 3x^2y^2} = \frac{40x^3 - 2y^3}{12ay^3 - 3x^2y^2}$$

Putting $x = y\left(\frac{1}{5}\right)^{\frac{1}{3}}$ values from (A)

$$2y^5\left(\frac{1}{5}\right)^{\frac{5}{3}} - 3a \cdot y^4 - y^5\left(\frac{1}{5}\right)^{\frac{2}{3}} = 0.$$

$$2y\left(\frac{1}{5}\right)^{\frac{5}{3}} - 3a - y\left(\frac{1}{5}\right)^{\frac{2}{3}} = 0 \quad y\left\{2\left(\frac{1}{5}\right)^{\frac{5}{3}} - 5\left(\frac{1}{5}\right)^{\frac{5}{3}}\right\} = 3a$$

$$3y \cdot \left\{\frac{1}{5}\right\}^{\frac{5}{3}} = 3a.$$

$$y = a \cdot (5)^{\frac{5}{3}}.$$

$$x = y \cdot \left(\frac{1}{5}\right)^{\frac{1}{3}} = a(5)^{\frac{4}{3}}.$$

$$\frac{d^2y}{dx^2} = + \frac{40x^3 - 2y^3}{12xy^3 - 3x^4y^2} = \frac{2}{a}.$$

$\frac{2}{5} \cdot y \cdot \left(\frac{1}{5}\right)^{\frac{5}{3}} = a \cdot$

$\frac{2}{5} \cdot y \cdot : a \cdot 5^{\frac{5}{3}}$

$u = (x-1)^4 (x+2)^3$.

$\frac{du}{dx} = 4(x-1)^3(x+2)^3 + 3(x-1)^4(x+2)^2$.

$= 4x+8\{(x-1)^3(x+2)^2 + 3x-3(x-1)^3(x+2)^2$

$= 7x+5\{(x-1)^3(x+2)^2 \cdot 2$ $= 0$ if $x = -\frac{5}{7}$ or $x = 1$ or $x = -2$

$\frac{d^2u}{dx^2} = 7\{(x-1)^3(x+2)^2\} + 3(x-1)^2(x+2)^2(7x+5)$

$+ 2(x-1)^3(x+2)(7x+5)$.

$= 7(x-1)(x+2)\{(x-1)^2(x+2)$

$+ 7x+5\{3x+6+2x-2)(x+2)(x-1)^2$.

$= \{7(x^2-1)(x+2) + (7x+5)(5x+4)\{(x+2)(x-1)^2 = 0$ if $x = -2$

$= 0$ if $x = 1$.

Value when $x = -\frac{5}{7}$.

$\therefore u = \frac{12^4}{7^4} \cdot \left(\frac{9}{7}\right)^3$

$= \frac{12^4 \times 9^3}{7^7}$ a maximum when $x = -\frac{5}{7}$.

$$u = x^4 + 3x^3 - 7x^2 - 27x - 18$$

$$v = x^4 - 3x^3 - 7x^2 + 27x - 18.$$

$$\frac{u_x}{v_x} = \frac{4x^3 + 9x^2 - 14x - 27}{4x^3 - 9x^2 - 14x + 27} = \frac{189 - 69}{135 - 123}$$

$$= \frac{120}{12} = 10.$$

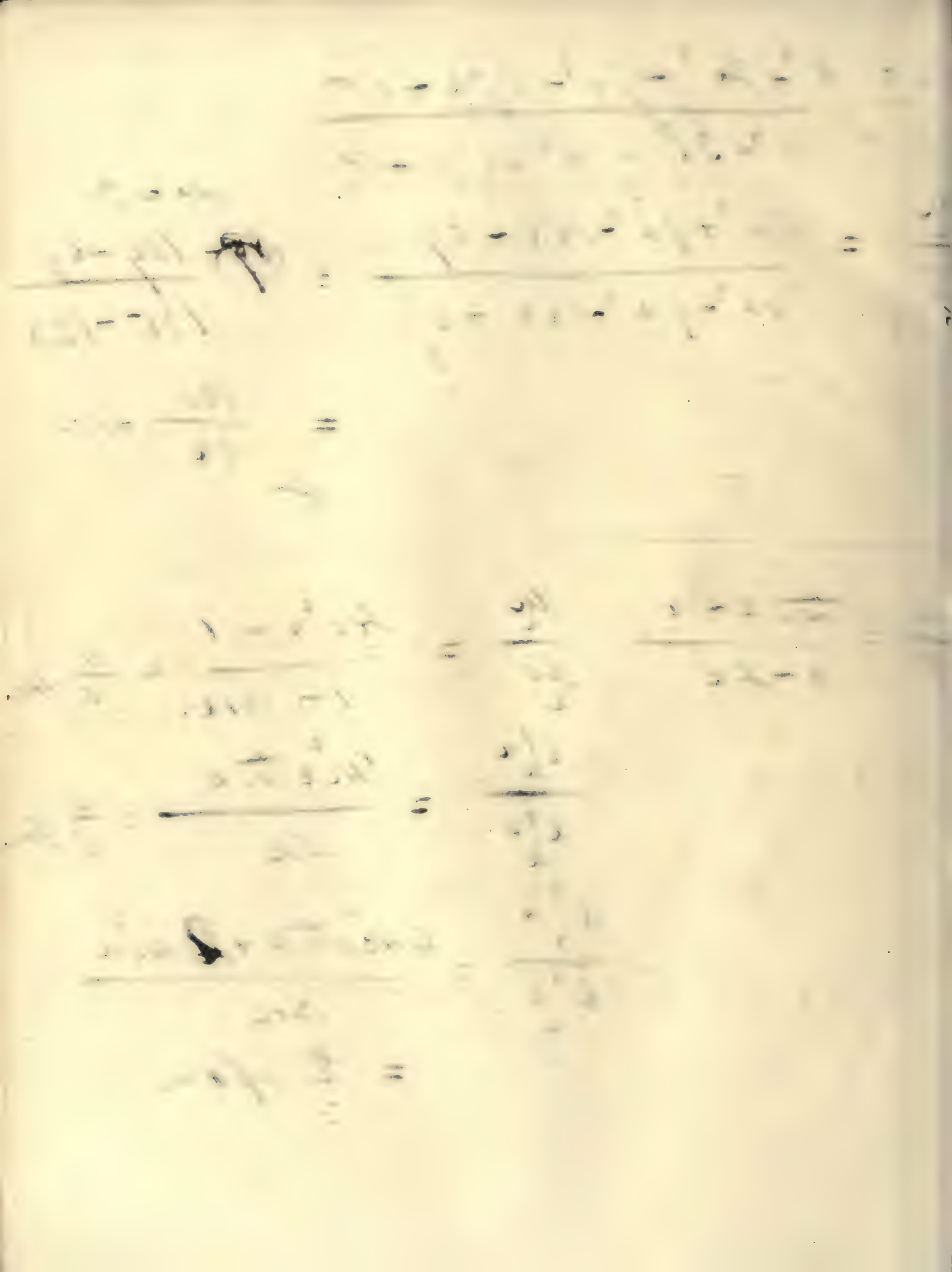
$$\frac{u}{v} = \frac{\tan x - x}{x - \sin x}$$

$$\frac{du_x}{dv_x} = \frac{\sec^2 x - 1}{1 - \cos x} = \frac{0}{0} \text{ for } x = 0$$

$$\frac{d_x^2 u}{d_x^2 v} = \frac{2\sec^2 x \tan x}{\sin x} = \frac{0}{0} \text{ for } x = 0$$

$$\frac{d_x^3 u}{d_x^3 v} = \frac{4\sec^2 x \tan x + 2\sec^2 x}{\cos x}$$

$$= 2 \text{ when } x = 0$$



the second part after num. number when $a = \cos x$.
the first part neg.

$\therefore$ u is a max. if $x = \cos x$.

$$u = x^{\frac{1}{x}}$$

$$\log_\xi u = \frac{1}{x} \log_\xi x.$$

$$\frac{du}{u} = \frac{1}{x^2} - \frac{1}{x^2} \log_\xi x.$$

$$du = \frac{x^{\frac{1}{x}}}{x^2} \left\{ 1 - \log_\xi x \right.$$

$$= x^{\frac{1}{x}-2} \left\{ 1 - \log_\xi x \right) = 0 \text{ if } \log_\xi x = 1$$

or $x = \xi$.

$$d^2u = x^{\frac{1}{x}-2} + -\frac{1}{x} \log_\xi - \frac{2x^{\frac{1}{x}-2}}{x}$$

$$= -3x^{\frac{1}{x}-3} \text{ which is neg. if } x = \xi.$$

$\therefore$ $u = x^{\frac{1}{x}} = \xi^{\frac{1}{\xi}}$ a maximum.

$$v = \frac{1}{x} + \tan x$$

$$\frac{dv}{dx} = -\frac{1}{x^2} + \frac{1}{\cos^2 x} = 0 \Rightarrow x = \pm \cos x$$

$$\frac{d^2v}{dx^2} = \frac{2}{x^3} + \frac{2\sin x}{\cos^3 x}$$

$$= \frac{2 + 2\sin x}{\cos^3 x}$$

$\therefore$ v is a min.

Now is a max.

$$\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}.$$

If $x = a+h$. $\therefore \sqrt{x} = \sqrt{a}\cdot\frac{1}{2}\frac{h}{\sqrt{a}} + \alpha$.

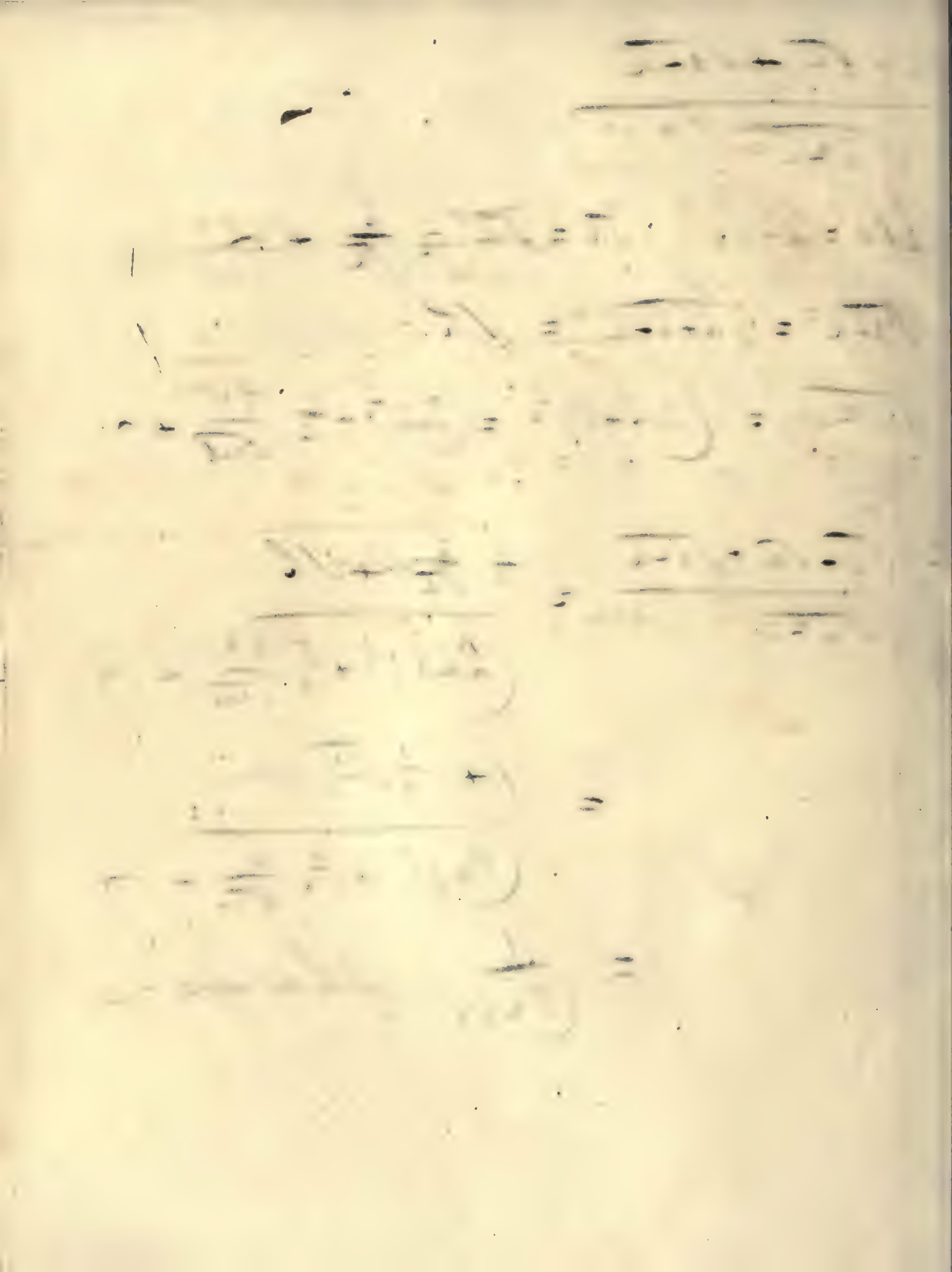
$$\sqrt{x-a} = \sqrt{a+h-a} = \sqrt{h}.$$

$$\sqrt{x^2-a^2} = (2ah+h^2)^{\frac{1}{2}} = (2ah)^{\frac{1}{2}} + \frac{1}{2}\frac{h^2}{\sqrt{2ah}} + \alpha.$$

$$\therefore \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}} = \frac{\frac{1}{2}\frac{h}{\sqrt{a}}\alpha + \sqrt{h}}{(2ah)^{\frac{1}{2}} + \frac{1}{2}\frac{h^2}{\sqrt{2ah}} + \alpha}$$

$$= \frac{1 + \frac{1}{2}\sqrt{\frac{h}{a}}\alpha}{(2a)^{\frac{1}{2}} + \frac{1}{2}\frac{h}{\sqrt{2a}} + \alpha}$$

$$= \frac{1}{(2a)^{\frac{1}{2}}} \text{ if } h = 0 \text{ or } x = a$$



$x = a(1 - \cos\theta)$.

$y = a(\theta + n\theta)$.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{a(\cos\theta + n)}{a\,d\theta} = \frac{\cos\theta + n}{d\theta}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{\cos\theta + n}{d\theta}\right\} = \frac{d}{d\theta}\left(\frac{\cos\theta + n}{d\theta}\right) \cdot \frac{1}{a\,d\theta}$$

$$= \frac{-1/4\, n\cos\theta}{a\,d^3\theta} = 0 \text{ if } \cos\theta = \frac{1}{n}.$$

$\therefore$ there are ∞ pts. of cons. flex.

Ex. $e^y = \frac{e^x+1}{e^x-1}$

$\therefore y = \log_e(e^x+1) - \log_e(e^x-1)$

$\frac{dy}{dx} = \frac{e^x}{e^x+1} - \frac{e^x}{e^x-1}$

$= \frac{e^{2x} - e^x - e^{2x} - e^x}{e^{2x}-1} = \frac{-2e^x}{e^{2x}-1}$

$\therefore ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \int \frac{\sqrt{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}}{(e^{2x}-1)^{1/2}}\,dx$

$= \frac{e^{2x}+1}{e^{2x}-1}$

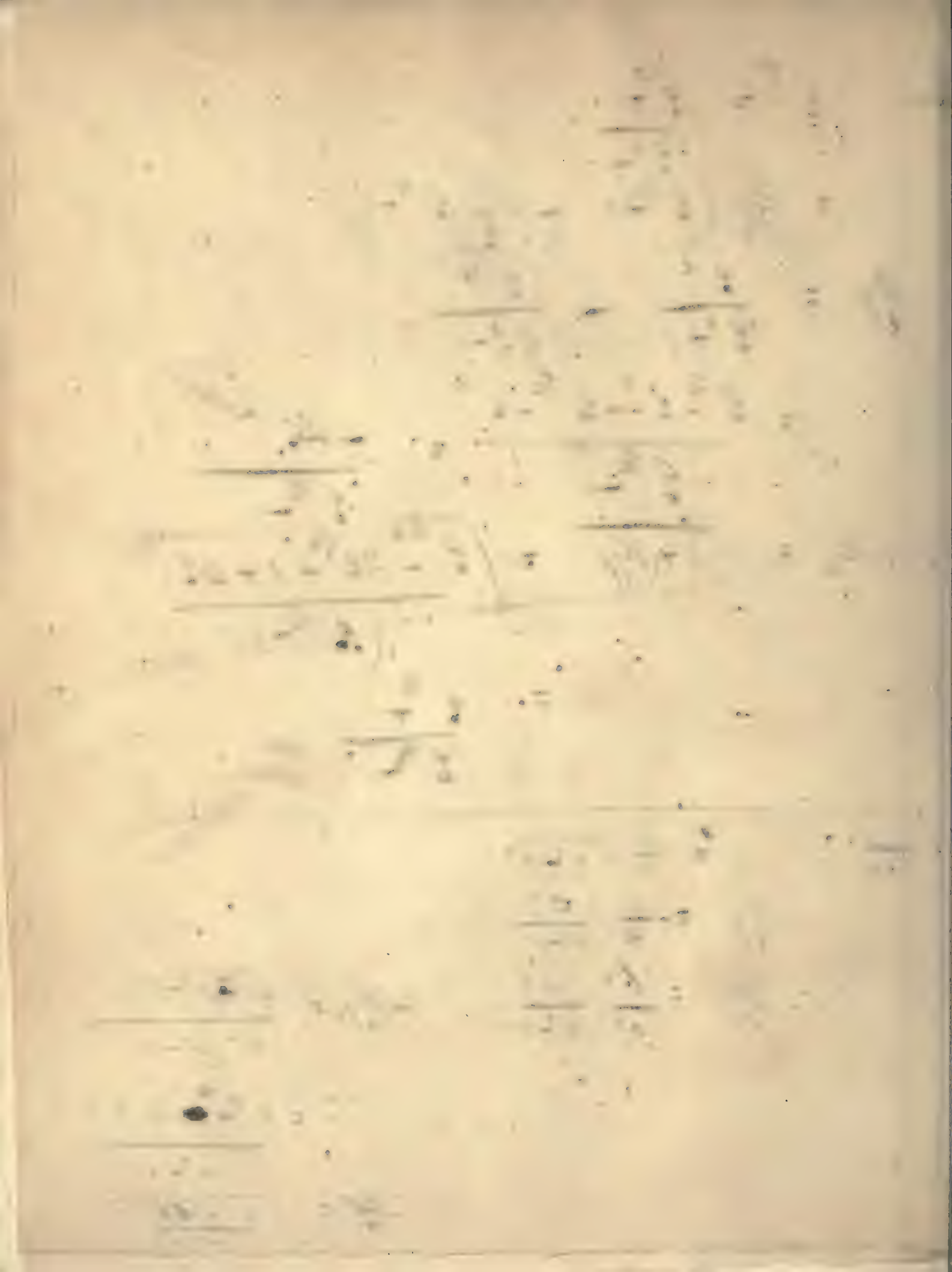
Ex. $y = \frac{b}{a}\sqrt{a^2-x^2}$

$\frac{dy}{dx} = -\frac{b}{a}\cdot\frac{x}{\sqrt{a^2-x^2}}$

$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{b^2}{a^2}\cdot\frac{x^2}{a^2-x^2}$ $\therefore 1+\left(\frac{dy}{dx}\right)^2 = \frac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}$

$= \frac{a^2+e^2x^2}{a^2-x^2}$

$\therefore ds = \int \frac{\sqrt{a^2-e^2x^2}}{a^2-x^2}\,dx$



(1) $\int_x \frac{x\,dx}{(a+bx)^{\frac{1}{2}}} = \int_x x(a+bx)^{-\frac{1}{2}}$.

Let $a+bx = z \therefore d_x^z = b$ also $x = \frac{z-a}{b}$.

$\therefore x = \frac{(z-a)}{b^2} d_x z$. $\therefore \int_x x(a+bx)^{-\frac{1}{2}} = \int_x z^{-\frac{1}{2}} \left(\frac{z-a}{b^2}\right) d_x^z$.

$= \frac{1}{b^2} \int_z \left(z^{\frac{1}{2}} - a z^{-\frac{1}{2}}\right) = \frac{1}{b^2} \left\{\frac{2}{3} z^{\frac{3}{2}} - 2az^{\frac{1}{2}}\right) = \frac{2z^{\frac{1}{2}}}{3b^2} \{z - 3a)$

$= \frac{2(a+bx)^{\frac{1}{2}}}{3b^2} \{bx - 2a)$

(2) $\int_x \frac{x^3}{(x-1)^{\frac{1}{2}}} = \int_x x^3 (x-1)^{-\frac{1}{2}}$.

Let $x-1 = z \quad \therefore d_x^z = 1 \quad x = z+1 = (z+1) d_x^z$.

$\int_x x^3 (x-1)^{-\frac{1}{2}} = \int_z z^{-\frac{1}{2}} (z^3 + 3z^2 + 3z + 1) = \int_z z^{\frac{5}{2}} + 3z^{\frac{3}{2}} + 3z^{-\frac{1}{2}} + z^{-\frac{1}{2}}$.

$= \frac{2}{7} z^{\frac{7}{2}} - \frac{6}{5} z^{\frac{5}{2}} + \frac{6}{3} z^{\frac{3}{2}} - 2z^{\frac{1}{2}}$.

$= 2z^{\frac{1}{2}} \left\{\frac{z^3}{7} + \frac{3}{5} z^2 + z + 1\right) = 2(x-1)^{\frac{1}{2}} \left\{\frac{(x-1)^3}{7} + \frac{3}{5}(x-1)^2 + x\right\}$

$$\int_x \frac{1}{(1-x^2)^{\frac{3}{2}}} = \int_x \frac{1}{x^3\left(x^{-2}-1\right)^{\frac{3}{2}}} = \int_x x^{-3}\left(x^{-2}-1\right)^{-\frac{3}{2}}.$$

$$= -\frac{x^{-3}\left(x^{-2}-1\right)^{-\frac{1}{2}}}{\frac{1}{2}\times -2x^{-3}} = \left(x^{-2}-1\right)^{-\frac{1}{2}} = \frac{1}{\left(x^{-2}-1\right)^{\frac{1}{2}}} = \frac{x}{\left(1-x^2\right)^{\frac{1}{2}}}$$

$$\int_x \frac{1}{(a+bx^2)^{\frac{3}{2}}} = \int_x \frac{1}{x^3\left(ax^{-2}+b\right)^{\frac{3}{2}}} = \int_x x^{-3}\left(ax^{-2}+b\right)^{-\frac{3}{2}}$$

$$= \frac{x^{-3}\left(ax^{-2}+b\right)^{-\frac{1}{2}}}{-\frac{1}{2}\times -2ax^{-3}} = \frac{\left(ax^{-2}+b\right)^{-\frac{1}{2}}}{+a} = +\frac{1}{a\left(ax^{-2}+b\right)^{\frac{1}{2}}} = \frac{x}{a(a+bx^2)^{\frac{1}{2}}}$$

$$\int_x \frac{1}{x^2(1-x^2)^{\frac{1}{2}}} = \int_x \frac{1}{x^2\left(x^{-2}-1\right)^{\frac{1}{2}}} = \int_x x^{-2}\left(x^{-2}-1\right)^{-\frac{1}{2}}$$

$$= \frac{x^{-2}\left(x^{-2}-1\right)^{-\frac{1}{2}}}{\frac{1}{2}\times -2x^{-3}} = -\left(x^{-2}-1\right)^{\frac{1}{2}} = -\frac{\left(1-x^2\right)^{\frac{1}{2}}}{x}$$

$$\int_x \frac{1}{(a+bx^n)^{\frac{n+1}{n}}} = \int_x x^{-(n+1)}\left(ax^{-n}+b\right)^{-\frac{n+1}{n}}$$

$$= \frac{x^{-(n+1)}\cdot\left(ax^{-n}+b\right)^{-\left(\frac{n+1}{n}\right)}}{-\frac{1}{n}\times -nax^{-(n+1)}} = \frac{\left(ax^{-n}+b\right)^{-\frac{1}{n}}}{a} = \frac{x}{a\left(a+bx^n\right)^{\frac{1}{n}}}$$

$$\frac{1}{x^5+2x^3+3x^2} = \frac{1}{x^2(x+1)(x^2-x+3)} = \frac{A}{x^2}+\frac{B}{x}+\frac{C}{x+1}+\frac{D}{x^2-x+3}$$

$$1 = A(x+1)(x^2-x+3) + Bx(x+1)(x^2-x+3) + C\ldots$$

$x=-1$

$1 = 3A \Rightarrow A = \frac{1}{3}$

$$\frac{-x^3-\frac{2}{3}x}{x} = -\frac{1}{3}\left(x^2+2\right) = B(x+1)(x^2-x+3) + Cx(x+1)(x^2-x+3) + \ldots$$

$x=0$

$-\frac{2}{3} = 3B \Rightarrow B = -\frac{2}{9}$

$$\frac{x^2-6+2x^3+4x+6}{9} = \frac{2x^3-3x^2+4x}{9x} = C(x+1)(x^2-x+3) + D(x+1)$$

$$\frac{2x^2-3x+4}{9} = C(x^2-x+3) + D(x+1)$$

$x=-1$

$1 = C(5) \therefore C = \frac{1}{5}$

$$\frac{4x^2-15x+20-9x^2+9x-27}{45(x+1)} = \frac{x^2-6x-7}{45(x+1)} = \frac{x-7}{45}$$

$$\therefore \frac{1}{x^5+2x^3+3x^2} = \frac{1}{3}\int\frac{1}{x^2} - \frac{2}{9}\int\frac{1}{x} + \frac{1}{5}\int\frac{1}{x+1} + \int\frac{x-7}{45(x^2-x+3)}$$

$$= -\frac{1}{3}\frac{1}{x} - \frac{2}{9}\ln x + \frac{1}{5}\ln(x+1) + \frac{1}{90}\ln(x^2-x+3)$$

$$-\frac{13}{45\sqrt{11}}\tan^{-1}\frac{2x-1}{\sqrt{11}}$$

If the density of a fluid vary directly as the depth. a body of uniform density will float in it when the density of the fluid at the depth of the c. of g. of the body is the same as the density of the body.

Let $dx.dy.dz$ = volume of an element of the fluid displaced
z = its dist. below the surf.

Then since the density is a fn. of the depth we have

mass of the element $= \mu z.dx.dy.dz$.

$\therefore \iiint \mu z \int dx \int dy \int dz\, z$ = weight of the fluid displaced

the integrals being taken between the proper limits.

Let $\bar{z}$ = depth of the c. of g. of the solid below the surf of the fluid. the axis of y being in the surf of the fluid then $\bar{z} = \dfrac{\int x \int y \int z\, z}{\int x \int y \int z\, 1}$. the limits being the same as before.

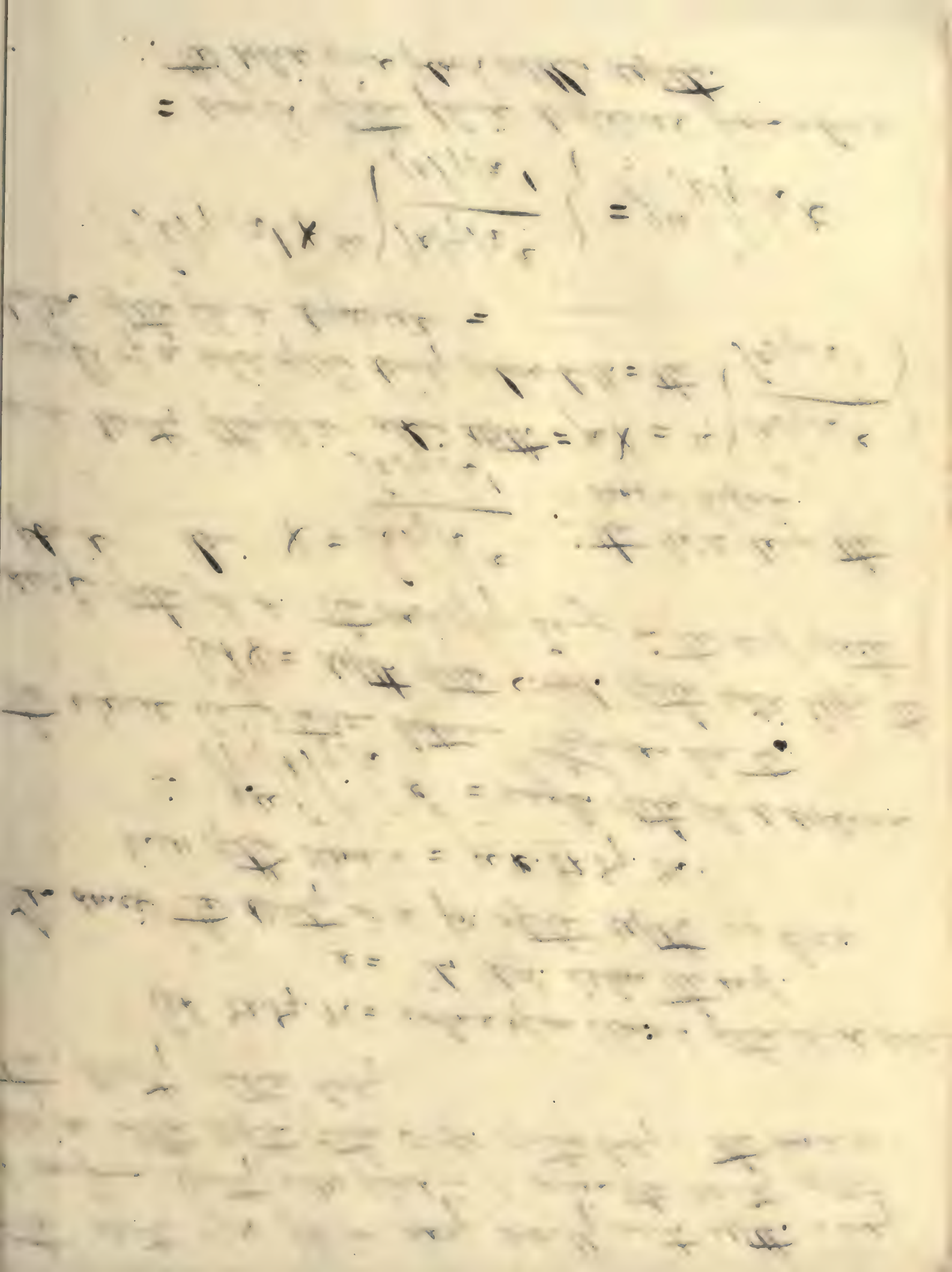
and density of the fluid at this depth $= \mu\bar{z} = \mu \left\{ \dfrac{\int x \int y \int z\, z}{\int x \int y \int z\, 1} \right\}$.

$\therefore$ weight of a solid of this density whose bulk = the bulk of the fluid displaced =

$$g \int x \int y \int z\, 1 \times \mu \left\{ \frac{\int x \int y \int z\, z}{\int x \int y \int z\, 1} \right\} = g\mu \int x \int y \int z\, z.$$

= weight of the fluid displaced from above.

$\therefore$ the solid will float at this depth.



$$\frac{1}{(1+x)\sqrt{1+x-x^2}}$$

Let $1+x = z \quad \therefore \quad dx = dz$.

$$\int_k \frac{1}{(1+x)\sqrt{1+x-x^2}} = \int_1 \frac{dz}{z\sqrt{z-(z-1)^2}}$$

$$= \int_z \frac{1}{z\sqrt{-1+3z-z^2}}$$

$$= -\int_y \frac{1}{\sqrt{-y^2+3y-1}} \qquad y = \frac{1}{1+z}.$$

$$= -\int_y \frac{dy\,.\left(y-\frac{3}{2}\right)}{\sqrt{\frac{5}{4}-\left(y-\frac{3}{2}\right)^2}}$$

$$= -\sin^{-1}\left(\frac{2y-\frac{3}{2}}{\sqrt{5}}\right).$$

$$= \pi - \sin^{-1}\frac{3x+1}{(x+1)\sqrt{5}}$$

$$u = \frac{1}{x + x^{\frac{2}{3}}} = \frac{1}{x^{\frac{2}{3}}\left(x^{\frac{1}{3}}+1\right)} = x^{-\frac{2}{3}}\left(x^{\frac{1}{3}}+1\right)^{-1}.$$

Let $x^{\frac{1}{3}} + 1 = z \quad \therefore \quad dz = \frac{1}{3} x^{-\frac{2}{3}}$

$$x = (z-1)^3.$$

$$\therefore x^{-\frac{2}{3}} = 3x^{\frac{1}{3}}\, dz.$$

$$\therefore \int_x u^{-\frac{2}{3}}\left(x^{\frac{1}{3}}+1\right)^{-1} = 3\int_x (z-1)^4 z^{-1}\, dz$$

$$= 3\left(z^3 - 4z^2 + 6z - 4 + z^{-1}\right).$$

$$= 3\left(\frac{1}{4}z^4 - \frac{4}{3}z^3 + 3z^2 - 4z + \log z\right).$$

$$u = \frac{6x^6 - 3x^5 + 6x - 2}{2x^4 + x^3} = \frac{(6x^5 + 2)(2x-1)}{x^3(2x+1)}$$

$$= \frac{6x^5 + 2}{x^3} = 6x^2 + 2x^{-3}.$$

$$\int_x u = 2x^3 - x^{-2}.$$

$$\int x\,(x^2+1)\cdot\sqrt{x^2+4}$$

$$= \int x\,(x^2+4)^{\frac{1}{2}} - 3\int x\,(x^2+4)^{\frac{1}{2}}$$

$$= x\,(x^2+4)^{\frac{3}{2}} - 3\int x^2\,(x^2+4)$$

$$= x(x^2+4)^{\frac{3}{2}} - 3\int (x^2+4)^{\frac{3}{2}} + 12\int x\,(x^2+4)$$

$$\therefore \int x\,(x^2+1)^{\frac{3}{2}} = \frac{1}{4}\,x(x^2+4)^{\frac{3}{2}} + 3\int x\,(x^2+4)$$

$$=$$

$$\therefore \int x\,(x^2+1)\sqrt{x^2+4} = \frac{1}{4}\,x\,(x^2+4)^{\frac{3}{2}}$$

$$f(x) = \frac{n}{x} \cdot \frac{\varphi(x+1)}{\varphi(x)}$$

$$\therefore f(x+1) = \frac{n}{x+1} \cdot \frac{\varphi(x+2)}{\varphi(x+1)}$$

$$\text{Cont } f(x) = \frac{n}{x + f(x+1)}$$

$$= \frac{n}{x + \frac{n}{x+1} \cdot \frac{\varphi(x+2)}{\varphi(x+1)}}$$

$$= \frac{(x+1)\varphi(x+1)}{x(x+1)\varphi(x+1) + n\varphi(x+2)}.$$

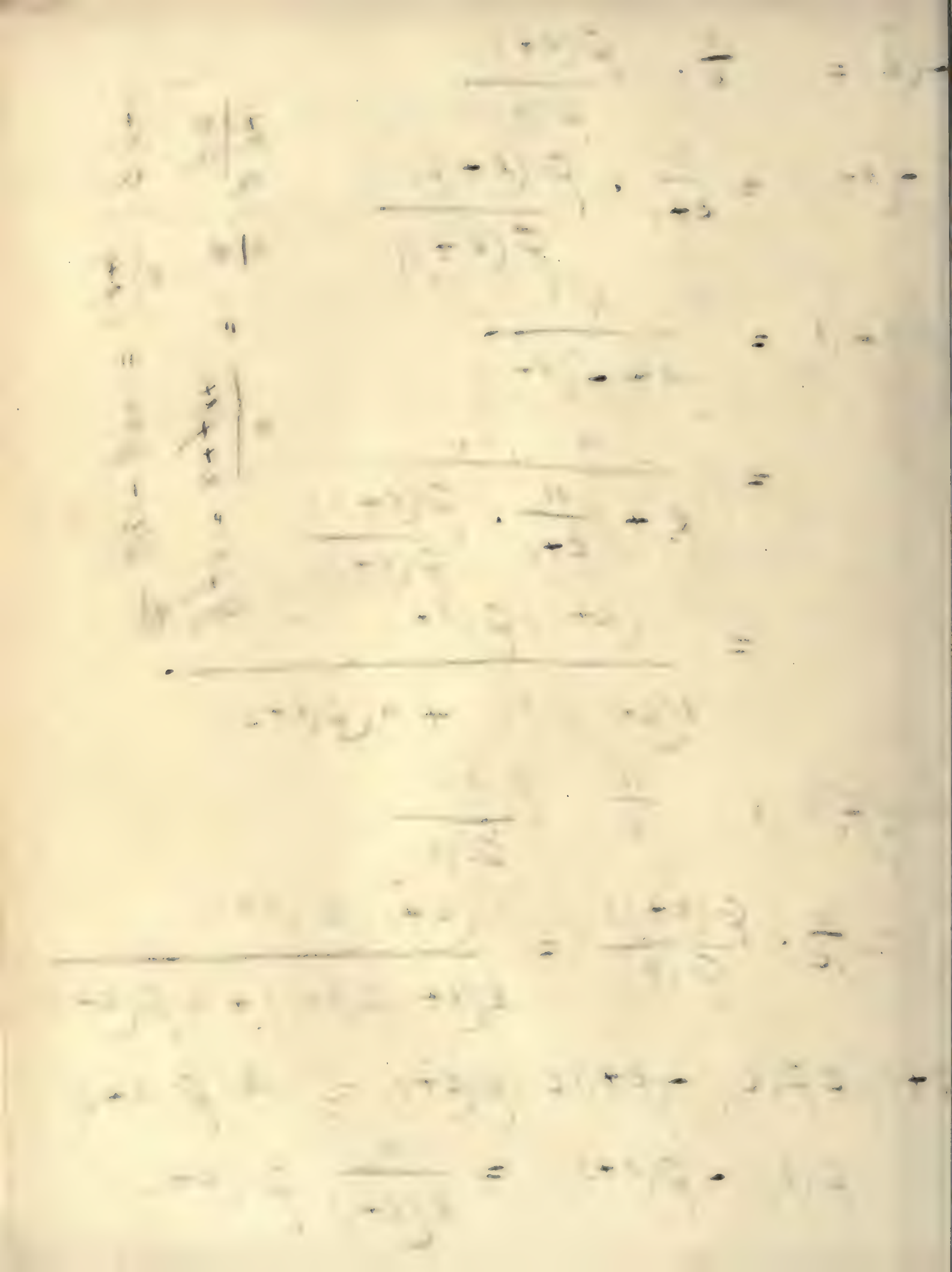
$$\text{But } f(x) = \frac{n}{x} \cdot \frac{\varphi(x+1)}{\varphi(x)}$$

$$\therefore \frac{n}{x} \cdot \frac{\varphi(x+1)}{\varphi(x)} = \frac{(x+1)\varphi(x+1)}{x(x+1)\varphi(x+1) + n\varphi(x+2)}$$

$$(x+1)\, x\, \varphi(x) - (x+1)x \cdot \varphi(x+1) = n \cdot \varphi(x+2)$$

$$\varphi(x) - \varphi(x+1) = \frac{n}{x(x+1)} \varphi(x+2)$$

$$\text{If } f(x+1) = \frac{n}{x+f(x+1)} \text{ then } \varphi(x) - \varphi(x+1) = \frac{n}{x(x+1)} \{\varphi(x+2)\}$$



$$d_z v = d_\rho v\, d_z \rho + d_\varphi v\, d_z \varphi$$

Now $y^2 + z^2 = \rho^2$

$$\therefore \rho\, d_z \rho = z$$

$$d_z \rho = \frac{z}{\rho} = \cos\varphi$$

$$= d_\rho v \cos\varphi - d_\varphi v \frac{\sin\varphi}{\rho}$$

Also $\tan\varphi = \frac{y}{z}$.

$$d_z \varphi = \frac{-\frac{y}{z^2}}{\frac{z^2+y^2}{z^2}} = -\frac{\sin\varphi}{\rho}$$

$$= d_\rho \left\{ d_\rho v \cos\varphi - d_\varphi v \frac{\sin\varphi}{\rho} \right\} \cos\varphi + d_\varphi \left\{ d_\rho v \cos\varphi - d_\varphi v \frac{\sin\varphi}{\rho} \right\} \left(-\frac{\sin\varphi}{\rho} \right)$$

$$= d_\rho^2 v \cos^2\varphi - d_\rho v\, d_\rho \cos\varphi \cdot \cos\varphi - d_\rho d_\varphi v \frac{\sin\varphi \cos\varphi}{\rho} - d_\varphi v\, d_\rho \frac{\sin\varphi}{\rho} \cdot \cos\varphi$$

$$- d_\varphi d_\rho v \frac{\cos\varphi \sin\varphi}{\rho} - d_\rho v\, d \frac{\cos\varphi}{\rho} \frac{\sin\varphi}{\rho} + d_\varphi^2 v \frac{\sin^2\varphi}{\rho^2} + d_\varphi v\, d_\varphi \frac{\sin\varphi}{\rho} \cdot \frac{\sin\varphi}{\rho}$$

$$= d_\rho^2 v \cos^2\varphi - 2 d_\rho d_\varphi v \frac{\sin\varphi \cos\varphi}{\rho} + 2 d_\varphi v \frac{\sin\varphi \cos\varphi}{\rho^2}$$

$$- d_\rho v \cdot \frac{\sin^2\varphi}{\rho}$$

$$= d_\rho^2 v \sin^2\varphi + d_\varphi v \frac{\sin^2\varphi}{\rho} \Big\} - 2 d_\varphi v \Big\}$$

$$- \frac{2 \sin\varphi \cos\varphi}{\rho^2} \Big\} \rho\, d_\rho d_\varphi v - d_\varphi v \Big\} - d_\rho v \frac{\sin^2\varphi}{\rho} \qquad (2)$$

Add together $= (1) + (2)$

$$d_y^2 v + d_z^2 v = d_\rho^2 v + d_\varphi^2 v \frac{1}{\rho^2} + \frac{1}{\rho} d_\rho v \qquad (3)$$

Also $d_x^2 v + d_\rho^2 v = d_r^2 v + d_\theta^2 v \frac{1}{r^2} + \frac{1}{r} d_r v \qquad (4)$.

Laplace Equation in cylindrical coordinates (3). (4).

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial \rho^2}\frac{1}{\rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial \theta^2}\frac{1}{r^2} + \frac{1}{r}\frac{\partial V}{\partial r} = 0$$

But $\frac{1}{\rho}\frac{\partial V}{\partial \rho} = \frac{1}{r\sin\theta}\frac{\partial V}{\partial \rho} =$

$$= \frac{1}{r\sin\theta}\left\{\frac{\partial V}{\partial r}\frac{\partial r}{\partial \rho} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial \rho}\right\} = \frac{1}{r\sin\theta}\left\{\frac{\partial V}{\partial r}\sin\theta + \frac{\partial V}{\partial \theta}\frac{\cos\theta}{r}\right\}$$

$$= \frac{\partial V}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta}$$

∴ By substitution we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \rho^2} + \frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial \theta^2}\frac{1}{r^2} + \frac{\partial V}{\partial r}\frac{2}{r} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} = 0$$

$$\therefore r^2\frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial \theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial \rho^2} + 2r\frac{\partial V}{\partial r} + \frac{\partial V}{\partial \theta}\cot\theta = 0$$

$$r\left(\frac{\partial}{\partial r} rV\right) = r\left\{r\frac{\partial V}{\partial r} + V\right\} \quad \therefore r\left(\frac{\partial^2 V}{\partial r^2} r\right) = r\left\{r\frac{\partial^2 V}{\partial r^2} + 2\frac{\partial V}{\partial r}\right\}$$

$$= r^2\frac{\partial^2 V}{\partial r^2} + 2r\frac{\partial V}{\partial r}.$$

$$\therefore \nabla = r^2\frac{\partial^2}{\partial r^2}(Vr) + \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial V}{\partial \theta}\cot\theta + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial \rho^2}$$

$$y^2 \frac{d^2u}{dy^2} + Hy \frac{du}{dy} + Bu = 0.$$

$$y = e^x \therefore dy = e^x = dx^2y.$$

$$\therefore y^2 \frac{d_x^2u \cdot e^x - du_x \cdot e^x}{(e^x)^3} + Hy \frac{du_x}{e^x} + Bu = 0.$$

$$\text{or } d_x^2u - du_x + H\, du_x + Bu = 0.$$

$$d_x^2u + (H-1)\, du_x + Bu = 0.$$

$$\frac{du}{dy} + \frac{u}{(1+y^2)^{\frac{1}{2}}} = a.$$

$$x = \log\{y + (1+y^2)^{\frac{1}{2}}\} \quad \therefore \frac{dx}{dy} = \frac{1 + \frac{y}{(1+y^2)^{\frac{1}{2}}}}{y + (1+y^2)^{\frac{1}{2}}}$$

But $\frac{du}{dy} = \frac{du}{dx}\frac{dx}{dy}$ $= \frac{1}{(1+y^2)^{\frac{1}{2}}}$.

$$\therefore \frac{\frac{du}{dx}}{(1+y^2)^{\frac{1}{2}}} + \frac{u}{(1+y^2)^{\frac{1}{2}}} = a.$$

$$\text{or } \frac{du}{dx} + u = a(1+y^2)^{\frac{1}{2}}.$$

Again $x = \log_e\{y + (1+y^2)^{\frac{1}{2}}\}$

$$\text{or } e^x = y + (1+y^2)^{\frac{1}{2}}$$

$$\left(e^x - y\right)^2 = 1 + y^2 = e^{2x} - 2e^x y + y^2$$

$$\therefore e^{2x} - 2e^x y = 1$$

$$e^{2x} - 2e^x y + y^2 \qquad \therefore y = \frac{1 - e^{2x}}{2e^x} = \frac{e^{-x} - e^x}{2}.$$

$$\text{or } y^2 = \frac{e^{-2x} - 2 + e^{2x}}{4} \quad \therefore y^2 + 1 = \frac{(e^{-x} + e^x)^2}{4} \quad \text{or } (1+y^2)^{\frac{1}{2}} = \frac{1}{2}(e^x + e^{-x})$$

$$\therefore \frac{du}{dx} + u = \frac{a}{2}\{e^x + e^{-x}\}$$

$$u = \frac{y^3 + x^3}{y - x}. \qquad \frac{du}{dx} = \frac{(y-x)3x^2 + (y^3+x^3)}{(y-x)^2}$$

$$= \frac{3x^2y - 3x^3 + y^3 + x^3}{(y-x)^2}.$$

$$\therefore x\frac{du}{dx} = \frac{3x^3y - 2x^4 + y^3x}{(y-x)^2}$$

$$y\frac{du}{dy} = y\left\{\frac{(y-x)3y^2 - (y^3+x^3)}{(y-x)^2}\right\}.$$

$$= \frac{3y^4 - 3xy^3 - y^4 - x^3y}{(y-x)^2}.$$

$$\therefore x\frac{du}{dx} + y\frac{du}{dy} = \frac{2y^4 - 2xy^3 + 2x^3y - 2x^4}{(y-x)^2}$$

$$= \frac{2y^3(y-x) + 2x^3(y-x)}{(y-x)^2}$$

$$= \frac{2(y^3+x^3)}{y-x}.$$

$uv = b^{ax} x^n$.

$d_x^r(uv) = u d_x^r v + r \frac{du}{dx} d_x^{r-1} v + \frac{r(r-1)}{1\cdot 2} d_x^2 u \, d_x^{r-2} v + \cdots$

$\because x^n = v \therefore d_x^r v = n(n-1)\cdots(n-r+1) x^{n-r}$.

$\therefore d_x^r(uv) = n(n-1)\cdots(n-r+1)\left\{ b^{ax} + \frac{ar}{n-r+1} b^{ax} + \frac{a^2 r(r-1)}{2(n-r+1)(n-r+2)} b^{ax} + \cdots \right\}$

(5). $uv = x^n b^{ax}$.

$\because v = b^{ax} \therefore d_x^r v = (-1)^{r-1} \cdot 1\cdot 2 \cdots (r-1) x^{-r}$

$\therefore d_x^r(uv) = (-1)^{r-1} \cdot 1\cdot 2 \cdots (r-1) x^{-r}\left\{ x^n + \frac{nr}{r-1} x^n + \frac{r(r-1)}{(1\cdot 2)^2} \frac{n(n-1)}{(r-1)(r-2)} x^n + \cdots \right\}$.

$= x^{n-r} \cdot (-1)^{r-1} 1\cdot 2 \cdots (r-1)\left\{ 1 + \frac{nr}{r-1} + \frac{r(r-1)}{(1\cdot 2)^2} \frac{n(n-1)}{(r-1)(r-2)} + \cdots \right\}$.

(6) $(uv) = \frac{(a+x)^n}{(c+x)^n}$. $\because v = (a+x)^n$.

$d_x^r(uv) = n(n-1)\cdots(n-r+1)(a+x)^{n-r}\left\{ \frac{1}{(c+x)^n} + \frac{r(a+x)}{n-r+1} \times \frac{-n}{(c+x)^{n+1}} \right.$

$\left. + \frac{r(r-1)}{2(n-r+1)(n-r+2)} \cdot \frac{(a+x)^2}{1\cdot 2} \cdot \frac{n(n+1)}{(c+x)^{n+2}} - \frac{r(r-1)(r-2)}{(1\cdot 2\cdot 3)^2 (n-r+1)(n-r+2)(n-r+3)} \frac{(a+x)^3 \cdot n(n+1)(n+2)}{(c+x)^{n+3}} \right.$

$u = \cos nx$.

$\frac{du}{dx} = -n \sin nx = n \cos\left(\frac{\pi}{2} + nx\right)$.

$\frac{d^r u}{dx^r} = n^r \cdot \cos\left(\frac{r\pi}{2} + nx\right)$.

$u = e^{x\cos\theta} \cos(x \sin\theta)$.

$\frac{du}{dx} = e^{x\cos\theta} \cdot \left(\cos\theta \cos(x\sin\theta) - \sin(x\sin\theta)\sin\theta\right)$

$= e^{x\cos\theta} \cos\{\theta + x\sin\theta\}$.

$\frac{d^2 u}{dx^2} = e^{x\cos\theta} \cdot \{\cos\theta \cos(\theta + x\sin\theta) - \sin\theta \sin(\theta + x\sin\theta)\}$

$= e^{x\cos\theta} \cos(2\theta + x\sin\theta)$

$\frac{d^r u}{dx^r} = e^{x\cos\theta} \cos(r\theta + x\sin\theta)$.

(c). $u = e^{ax} \cos nx$.

$\frac{du}{dx} = e^{ax}(a\cos nx\theta - n\sin nx)$

$= e^{ax} \cdot \cos(\theta + nx)$, if $\theta = \tan^{-1}\frac{a}{n}$.

$\frac{d^2 u}{dx^2} = e^{ax} \cdot a\cos(\theta + nx) - n\sin(\theta + nx)$

$= e^{ax} \cdot \cos(2\theta + nx)$ —

$\frac{d^m u}{dx^m} = e^{ax} \cdot \cos(n\theta + nx)$

$$u = \frac{1}{x^n}$$

Change x into $x+h$ and the expression becomes.

$$u' = \frac{1}{(x+h)^n} = (x+h)^{-n}.$$

$$= x^{-n} - n.x^{-(n+1)}h + \frac{n(n+1)}{1.2.}x^{-(n+2)}h^2 - \&c.$$

the coeff. of h^r is

$$\frac{n(n+1)\ldots(n+r-1)}{\lfloor r}.x^{-(n+r)}$$

$$\therefore d_x^r u = n(n+1)\ldots(n+r-1).x^{-(n+r)}.$$

$u = a^x$. $d_x u = a^x \log_e a$. $d_x^2 u = a^x (\log_e a)^2$

$$\therefore d_x^n u = a^x.(\log_e a)^n.$$

$u = e^{nx}$. $d_x u = n.e^{nx}$. $d_x^2 u = n^2 e^{nx}$.

$$\therefore d_x^n u = n^n.e^{nx}.$$

$u = \sin nx$. $d_x u = n\cos nx = n\sin\left(\frac{\pi}{2}+nx\right)$

$$d_x^2 u = d_x\, n\sin\left(\frac{\pi}{2}+nx\right) = n^2\cos\left(\frac{\pi}{2}+nx\right)$$

$$= n^2\sin\left(\pi+nx\right)$$

$$\therefore d_x^n u = n^n.\sin\left(\frac{n\pi}{2}+nx\right).$$

$y^n \log y = ax$.

$y^n \log y - ax = 0 \quad = u$.

$\frac{du}{dy} = n y^{n-1} \log y + y^{n-1} = y^{n-1}\{1 + n \log y\}$

$\frac{du}{dx} = -a$.

$\therefore \frac{dy}{dx} = \frac{a}{y^{n-1}\{1 + n \log y\}}$.

$\log_m y = 1 + x \log_m y$.

$\log_m y - 1 - x \log_m y = u = 0$.

$\frac{du}{dy} = \log \sec^2 y - x \cos y$

$\frac{du}{dx} = -\log_m y \quad \therefore \frac{dy}{dx} = \frac{\log_m y}{\log \sec^2 y - x \cos y}$

$= \frac{\cos^2 y \cdot \log y}{1 - x \cdot (\cos^3 y)}$.

$\log \tan \frac{y}{2} - \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} = 0 = u$.

$\frac{du}{dy} = \frac{1}{2} \log \sec^2 \frac{y}{2}$. $\frac{du}{dx} = -\frac{1}{2}\left\{\frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} + \frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}\right\} \div 1+x$

$= -\frac{1}{2}\left\{\frac{1+x+1-x}{(1-x^2)^{\frac{1}{2}}(1+x)}\right\} = -\frac{1}{(1-x^2)^{\frac{1}{2}}(1+x)}$

$\therefore \frac{dy}{dx} = \frac{-2}{(1-x^2)^{\frac{1}{2}}(1+x)\log \sec^2\left(\frac{y}{2}\right)}$

But $\log \sec^2 \frac{y}{2} = 1 + \frac{1-x}{1+x} = \frac{1+x+1-x}{1+x} = \frac{2}{1+x}$

$\therefore \frac{dy}{dx} = \frac{-1}{(1-x^2)^{\frac{1}{2}}}$

Exp. 9. p. 114.

$u = al + bm + cn.$

$du = adl + bdm + cdn. = 0$

$\therefore \lambda adl + \lambda bdm + \lambda cdn = 0.$

also $l^2 + m^2 + n^2 = 1$

$\therefore ldl + mdm + ndn = 0.$

Equate to zero the coeffts of $dl.dm.dn.$

$\therefore (\lambda a + l) = 0 \cdot (\lambda b + m) = 0, (\lambda c + n) = 0.$

from which values immediately

$$\frac{l}{a} = -\lambda = \frac{m}{b} = \frac{n}{c}$$

$$\therefore l = \frac{a}{b} m, \quad n = \frac{c}{b} m.$$

but $l^2 + m^2 + n^2 = 1$

$$\therefore \left(\frac{a^2}{b^2} + \frac{c^2}{b^2} + 1\right) m^2 = 1$$

$$\therefore m = \frac{b^2}{(a^2 + c^2 + b^2)} = \frac{b^2}{a^2 + c^2 + b^2}$$

$$\therefore m = \frac{b}{(a^2 + c^2 + b^2)^{\frac{1}{2}}}$$

$$n = \frac{c}{(a^2 + c^2 + b^2)^{\frac{1}{2}}}$$

$$l = \frac{a}{(a^2 + c^2 + b^2)^{\frac{1}{2}}}.$$

$u = \cos x \cdot \cos y \cdot \cos z$. (1)

$x + y + z = \pi$. (2)

from (1) $\log u = \log \cos x + \log \cos y + \log \cos z$.

$\therefore du = -dx \tan x - dy \tan y - dz \tan z$. (3)

from (2) $dx + dz + dz = 0$. (4)

Multiply (3) by λ, and add (4), equating to zero the coeffs of $dx . dy . dz$

$\therefore (1 - \lambda \tan x)\, dx = 0$

$(1 - \lambda \tan y)\, dy = 0$

$(1 - \lambda \tan z)\, dz = 0$.

from which values $\frac{1}{\tan x} = \lambda = \frac{1}{\tan y} = \frac{1}{\tan z}$

$\therefore \tan x = \tan y = \tan z$

$x = y = z$.

but $x + y + z = \pi \therefore x = y = z = \frac{\pi}{3}$.

$\cos x = \cos y = \cos z = \frac{1}{2}$

$\therefore u = \frac{1}{8}$ a max.

Let $x.y.z$ be the edges of the □ piped.

Then xyz = the surface volume.

$$u = 2(xy + zy + xz).$$

$$\therefore du = 2x\,dy + 2y\,dx + 2z\,dy + 2y\,dx + 2x\,dz + 2z\,dx = 0$$

$$\therefore \therefore (x+z)\,dy + (y+z)\,dx + (y+x)\,dz = 0 \quad (1)$$

But $xyz = a^3$ = volume.

$$\therefore \log x + \log y + \log z = 3\log a.$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \quad (2)$$

Multiply (1) by an indef. multiplier λ and add to (2) equating the coeffs of each diff. coeff. to Zero.

$$\therefore \lambda(y+z) + \frac{1}{x} = 0, \quad \lambda(x+z) + \frac{1}{y} = 0.$$

$$\therefore \lambda = \frac{1}{x(y+z)} = \frac{1}{y(x+z)} = \frac{1}{z(y+x)}$$

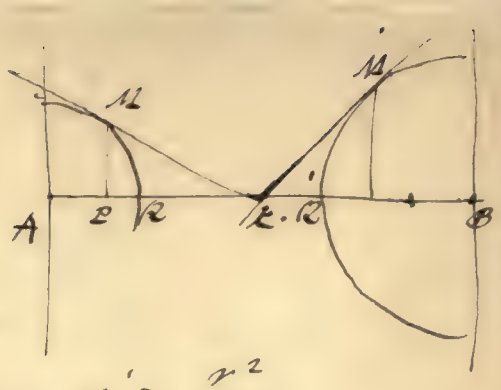
wherefore $x(y+z) = y(x+z) \quad \therefore \quad x = y$

$$x(y+z) = z(y+x) \quad \therefore \quad x = z.$$

$\therefore x = y = z$ and $xyz = a^3 \quad \therefore y = z = x = a$

and $6a^2$ is the area of the surface.

B. The centres of the Os. $AB = a$.
r' the radii of the Os. x the proposed point
$Ax = x \therefore xB = (a-x)$.
M. M' are taken at M. M' respectively.
Take A and B for the origin of the Os. and find the co-ord^{s} of M M'.



The $\perp^{r}$ to the tangents $yy' + ax' = r^2$ $\therefore$ if $y = 0$ $x' = \frac{r^2}{x'}$
$\therefore$ if k the centre of M $k = x = \frac{r^2}{x'}$.

Similarly $Rx = \frac{r'^2}{a-x}$.

area of surf. generated by the revolut. of M round the axis of x

$$= 2\pi a x' = \frac{2\pi r^3}{x}$$

area of surf. generated by the revolut. of M'M' — $= \frac{2\pi r'^3}{a-x}$

$\therefore$ whole surf $= 2\pi \left\{ \frac{r^3}{x} + \frac{r'^3}{(a-x)} \right\} = u$

$$\frac{du}{dx} = 2\pi \left\{ \frac{r'^3}{(a-x)^2} - \frac{r^3}{x^2} \right\} = 0 \text{ if } u = a \text{ max.}$$

$$\therefore \frac{r^3}{x^2} = \frac{r'^3}{a^2 - 2ax + x^2} \text{ or } (r^3 - r'^3)x^2 - 2ar^3x + r^3a^2 = 0$$

$$x^2 - \frac{2ar^3}{r^3 - r'^3}x + \frac{a^2r^6}{(r^3 - r'^3)^2} = \frac{a^2r^6 - (r^3 - r'^3)a^2r^3}{(r^3 - r'^3)^2} = \frac{a^2r'^3r^3}{(r^3 - r'^3)^2}$$

$$\therefore x = \frac{a}{r^3 - r'^3}\left\{ r^3 \pm \sqrt{r^3r'^3} \right\}$$

$$= \frac{ar^{\frac{3}{2}}}{r^{\frac{3}{2}} + r'^{\frac{3}{2}}} \left\{ \cdot \frac{r^{\frac{3}{2}} - r'^{\frac{3}{2}}}{r^{\frac{3}{2}} - r'^{\frac{3}{2}}} \right\}.$$

$$= \frac{ar^{\frac{3}{2}}}{r^{\frac{3}{2}} + r'^{\frac{3}{2}}}$$

$$d^2_x V + d^2_y V + d^2_z V = 0.$$

r. θ. φ.

$x = r\cos\theta$

$y = r\sin\theta\sin\varphi$

$z = r\sin\theta\cos\varphi$.

Let $\rho = r\sin\theta$

$\therefore y = \rho\sin\varphi$

$z = \rho\cos\varphi$.

$$d^2_y V = \quad -$$

$$d_y V = d_\rho V\, d_y \rho + d_\varphi V\, d_y \varphi.$$

$$= d_\rho V \sin\varphi + d_\varphi V \frac{\cos\varphi}{\rho}.$$

$$d^2_y V = d_\rho \left\{ d_\rho V \sin\varphi + d_\varphi V \frac{\cos\varphi}{\rho} \right\} \sin\varphi + d_\varphi \left\{ d_\rho V \sin\varphi + d_\varphi V \frac{\cos\varphi}{\rho} \right\} \frac{\cos\varphi}{\rho}$$

$$= d^2_\rho V \sin^2\varphi + d_\rho V\, d_\rho \sin\varphi \cdot \sin\varphi + d_\rho d_\varphi V \frac{\cos\varphi \sin\varphi}{\rho} + d_\varphi V\, d_\rho \frac{\cos\varphi}{\rho} \cdot \sin\varphi$$

$$+ d_\rho d_\varphi V \frac{\sin\varphi\cos\varphi}{\rho} + d_\varphi V \frac{\cos^2\varphi}{\rho} + d^2_\varphi V \frac{\cos^2\varphi}{\rho^2} + d_\varphi V\, d_\varphi \frac{\cos\varphi}{\rho} \cdot \frac{\cos\varphi}{\rho}$$

$$= d^2_\rho V \left\{ \sin^2\varphi + \frac{\cos^2\varphi}{\rho^2} \right\} + 2 d_\rho d_\varphi V \frac{\cos\varphi \sin\varphi}{\rho} + d_\varphi V \frac{\cos^2\varphi}{\rho} - 2\frac{\cos\varphi}{\rho^2} d_\varphi V$$

$$= d^2_\rho V \left\{ \sin^2\varphi + \frac{\cos^2\varphi}{\rho^2} \right\} + \frac{2\cos\varphi\sin\varphi}{\rho^2} \left\{ \rho\, d_\rho d_\varphi V - d_\varphi V \right\} \quad (1)$$

$$+ d_\rho V \cdot \frac{\cos^2\varphi}{\rho}$$

$$y = \rho\sin\varphi$$

$$y^2 + z^2 = \rho^2$$

$$y = \rho\, d_y \rho$$

$$\therefore d_y \rho = \frac{y}{\rho} = \sin\varphi$$

$$\varphi = \tan^{-1}\frac{y}{z}.$$

$$\therefore d_y \varphi = \frac{\frac{1}{z}}{\frac{y^2 + z^2}{z^2}} = \frac{z}{\rho^2} = \frac{\cos\varphi}{\rho}.$$

$u = ayz(c-z)$ $\therefore du = a(c-z)\,dy - ay\,dz = 0$

$u = bzx(a-x)$ $\therefore du = b(a-x)\,dz - bz.\,dx = 0$

$u = cxy(b-y)$ $\therefore du = c(b-y)\,dx - cx\,dy = 0$

Collecting together the coeffts of $dx.\ dy.\ dz.$

$\therefore a(c-z)$ $\therefore (c(b-y) - bz)\,dx = 0$ (1)

$(a(c-z) - cx)\,dy = 0$ (2)

$(b(a-x) - ay)\,dz = 0$ (3)

From these 3 we eliminate the values of $x.y.z.$

From (1) $z = \frac{c}{b}(b-y) = \frac{c}{a}(a-x)$ $by = ax$ (4)

$\therefore ab - ay = ab - bx$

$x = \frac{a}{b}y.$

From (3). $b(a-x) - abx = 0$ $\therefore x = \frac{a}{2}.$

Similarly $y = \frac{b}{2}.$ $z = \frac{c}{2}.$

$\therefore u = \frac{ab}{2}.\frac{c}{2} = \frac{1}{4}(abc)$ a maximum.

$$\tan(x+h) - \tan x = \frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} = \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{\cos(x+h)\cos x}$$

$$= \frac{\sin x \cosh \cos x + \cos^2 x \sinh - \cos x \sin x \cosh + \sin^2 x \sinh}{\cos(x+h)\cos x}$$

$$= \frac{\sin h}{\cos^2 x \cosh - \cos x \sin x \sinh} = \frac{\sin h}{\cos^2 x \cosh \{1 - \tan x \tan h\}}$$

$$= \frac{\sec^2 x \tan h}{1 - \tan x \tan h}$$

$$= \sec^2 x \text{ if } x = 0.$$

$$\text{Let } z = \tan^{-1}(x+h) \quad \therefore \tan z = x+h.$$

$$z' = \tan^{-1} x \quad \therefore \tan z' = x.$$

$$\tan(z - z') = \frac{\tan z - \tan z'}{1 + \tan z \cdot \tan z'}$$

$$= \frac{x+h-x}{1 + (x+h)x}$$

$$= \frac{h}{1 + x^2 + hx}$$

$$\therefore \tan^{-1}(x+h) - \tan^{-1} x = \tan^{-1}\left(\frac{h}{1+x^2+hx}\right)$$

$$\tan^{-1}(x+h) - \tan^{-1} x = \tan^{-1} \frac{h}{1+x^2+xh}.$$

$$= \frac{h}{1+x^2+xh} \text{ if } h \text{ is small}$$

$$\therefore \frac{\tan^{-1}(x+h) - \tan^{-1} x}{h} = \frac{1}{1+x^2+xh}$$

$$\left\{ \frac{\tan^{-1}(x+h) - \tan^{-1} x}{h} \right\}_{h=0} = \frac{1}{1+x^2} = d_x \tan^{-1} x.$$

$$x^2 - 2ay - a^2 - b = 0.$$

$$2x - 2a\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{x}{a} \text{ or } a = \frac{x}{dy/dx}$$

$$2 - 2a\frac{d^2y}{dx^2} = 0.$$

$$2 - 2\frac{x}{dy/dx}\frac{d^2y}{dx^2} = 0 \quad \therefore \quad \frac{dy}{dx} - x\frac{d^2y}{dx^2} = 0$$

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0.$$

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0.$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{dy/dt}{\sin t}$$

$$\frac{d^2y}{dx^2} = \frac{1}{\left(\frac{dx}{dt}\right)^3}\left\{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right\}$$

$$= -\frac{1}{\sin^3 t}\left\{-\sin t\frac{d^2y}{dt^2} + \frac{dy}{dt}\cos t\right\}$$

also since $t = \cos^{-1} x$ or $x = \cos t$, whence

$$1 - x^2 = 1 - \cos^2 t = \sin^2 t$$

$$\therefore -\sin^2 t\left\{\frac{\frac{dy}{dt}\cos t - \sin t\frac{d^2y}{dt^2}}{\sin^3 t}\right\} + \frac{\cos t}{\sin t}\frac{dy}{dt} + y = 0$$

$$\text{or} \quad -\frac{dy}{dt}\cdot\frac{\cos t}{\sin t} + \frac{d^2y}{dt^2} + \frac{dy}{dt}\frac{\cos t}{\sin t} + y = 0$$

$$\frac{d^2y}{dt^2} + y = 0.$$

$t = e^x$.

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\, e^x = t\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(t\frac{dy}{dt}\right) = \frac{d}{dt}\left(t\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{d}{dt}\left(t\frac{dy}{dt}\right)t$$

$$= t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2}\right)$$

$$= \frac{d}{dt}\left(t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2}\right)\frac{dt}{dx} = \frac{d}{dt}\left(t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2}\right)t$$

$$= t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2} + 2t^2\frac{d^2y}{dt^2} + t^3\frac{d^3y}{dt^3}$$

$$= t\frac{dy}{dt} + 3t^2\frac{d^2y}{dt^2} + t^3\frac{d^3y}{dt^3}.$$

$(x^2-x+1)^{-m}$.

$\text{Let } I = x(x^2-x+1)^{-m+1}$

$\frac{d}{dx} I = (x^2-x+1)^{-m} \{(-m+1)x(2x-1) + (x^2-x+1)\}$

$(-m+1)(2x^2-x) + (x^2-x+1)$

$= 2(1-m)x^2(x^2-x+1)$

$= (x^2-x+1)^{-m} \{(-m+1)(2x^2-2x+2+x-2) + (x^2-x+1)$

$= (x^2-x+1)^{-m} \{(3-2m)(x^2-x+1) + x - 2$.

$= (x^2-x+1)^{-m+1}(3-2m) + x(-2)(x^2-x+1)^{-m}(1-m)$

$\int \frac{1}{(x^2-x+1)^2} = \frac{x}{2(x^2-x+1)} + \int \frac{x}{2(x^2-x+1)^2} - \frac{1}{2}\int \frac{1}{(x^2-x+1)}$

$\text{Let } \int \frac{1}{2}\frac{x}{(x^2-x+1)^2} = \frac{1}{4}\int \frac{2x-1}{(x^2-x+1)^2} + \frac{1}{4}\int \frac{1}{(x^2-x+1)^2}$

$\frac{3}{4}\int \frac{1}{(x^2-x+1)^2} = \frac{1}{2}\frac{x}{(x^2-x+1)} + \frac{1}{4}\frac{d_x(x^2-x+1)}{(x^2-x+1)^2} + \frac{1}{2}\int \frac{1}{x^2-x+1}$

$\frac{1}{(x^2-x+1)^2} = \frac{4}{6}\frac{x}{(x^2-x+1)} + \frac{1}{3}\int \frac{d_x(x^2-x+1)}{(x^2-x+1)^2} + \frac{2}{3}\int \frac{1}{x^2-x+1}$

$= \frac{1}{3}\left(\frac{2x-1}{(x^2-x+1)} - \int \frac{1}{x^2-x+1}\right) \frac{1}{3}\frac{1}{(x^2-x+1)}$

$= \frac{1}{3}\log_e(x^2-x+1) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) - \frac{1}{3}\frac{1}{x^2-x+1}$

$$\frac{x^2+3x-2}{(x^2-x+1)^2(x-1)^2} = \frac{Ax+B}{(x^2-x+1)^2} + \frac{A_1x+B_1}{x^2-x+1} + \frac{C}{(x-1)^2} + \frac{D}{x-1}$$

$$x^2+3x-2 = (Ax+B)(x-1)^2 + (A_1x+B_1)(x^2-x+1)(x-1)^2 + C(x^2-x+1)^2 + D(x^2-x+1)^2(x-1)$$

Let $x^2 = x-1$

$$x-3 = (Ax+B)(-x) = -A(x-1) - Bx = -(A+B)x + A \quad \therefore A=-3,\ B=-6$$

$$3x-2 = (-3x-6)(x^2-2x+1) + \text{etc.}$$

$$= -x^3 + 2x^2 - x - 3x^2 + 6x - 3 + \text{etc.}$$

$$x^2 - x = -3x^3 + 6x^2 - 3x - x^2 + 2x - 1 + \text{etc.}$$

$$\frac{4x^2+4x+1}{x^2-x+1} = 3x-1 = (A_1x+B_1)(x-1)^2 + C(x^2-x+1) + D(x^2-x+1)(x-1)$$

Let $x^2 = x-1$

$$x-1 = (A_1x+B_1)(x^2-2x+1) = (A_1x+B_1)(-x) = A(1-x) - Bx$$

$$= A1 - (A+B)x \quad \therefore A=-1,\ B=-2$$

$$x-1 = -(x+2)(x^2-2x+1) + \text{etc.}$$

$$= -x^3 + 2x^2 - x - 2x^2 + 4x - 2 + \text{etc.}$$

$$\frac{x+1}{x^2-x+1} : x+1 = C + D(x-1) = C + Dx - D \quad \therefore D=1,\ C=2$$

$$\frac{x^2+3x-2}{(x^2-x+1)^2(x-1)^2} = -\frac{3x+1}{(x^2-x+1)^2} - \frac{x+2}{x^2-x+1} + \frac{2}{(x-1)^2} + \frac{1}{x-1}$$

$$\text{Now } -\frac{3x+1}{(x^2-x+1)^2} = -\frac{3}{2}\frac{d_x(x^2-x+1)}{(x^2-x+1)^2} - \frac{5}{2}\frac{1}{(x^2-x+1)^2}$$

$$= -\frac{3}{2}\frac{d_x(x^2-x+1)}{(x^2-x+1)^2} - \frac{5}{2}\left\{\frac{2}{3}\frac{d}{dx}\left(\frac{x}{x^2-x+1}\right) + \frac{1}{3}\frac{d_x\,x^2-x+1}{(x^2-x+1)^2} + \frac{2}{3}\frac{1}{x^2-x+1}\right.$$

$$\frac{x^2+3x-2}{(x^2-x+1)^2(x-1)^2} = \left(-\frac{5}{6}+\frac{3}{2}\right)\int \frac{d_x\,x^2-x+1}{(x^2-x+1)^2} - \frac{5}{3}\frac{x}{(x^2-x+1)} + \int\left(\frac{10}{6}-2\right)\frac{1}{(x^2-x+1)} - \int\frac{a}{x^2-x+1}$$

$$= \frac{7}{3}\frac{1}{x^2-x+1} - \frac{5}{3}\frac{x}{x^2-x+1} - \frac{22}{3\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + \text{etc.}$$

$$-\tan^{-1}\frac{2x-1}{\sqrt{3}} - \frac{1}{2}\log(x^2-x+1) + \log_e x-1$$

$$= -\frac{5x-7}{3(x^2-x+1)} - \frac{25}{3^{\frac{3}{2}}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + \frac{2}{x-1} + \log_e\left\{\frac{x-1}{(x^2-x+1)^{\frac{1}{2}}}\right\}$$

$$a^2y - x = a \tan \frac{a+bx}{x}$$

$$a^2y - x - a \tan \frac{a+bx}{x} = 0$$

$$2xy + x^2 \frac{dy}{dx} - 1 + a\left(\sec^2 \frac{a+bx}{x}\right) \cdot \frac{a}{x^2} = 0$$

$$\sec^2\left(\frac{a+bx}{x}\right) = 1 + \tan^2\left(\frac{a+bx}{x}\right)$$

$$2xy + x^2 \frac{dy}{dx} - 1 + \frac{a^2}{x^2}\left\{1 + \frac{x^4y^2 - 2x^3y + x^2}{a^2}\right\} = 0$$

$$2xy + x^2 \frac{dy}{dx} - 1 + \frac{1}{x^2}\left\{a^2 + x^4y^2 - 2x^3y + x^2\right\} = 0$$

$$x^2 \frac{dy}{dx} + \frac{a^2}{x^2} + x^2y^2 = 0$$

$$\frac{dy}{dx} + y^2 + \frac{a^2}{x^4} = 0.$$

$y = a^{\tan^{-1}x}$.

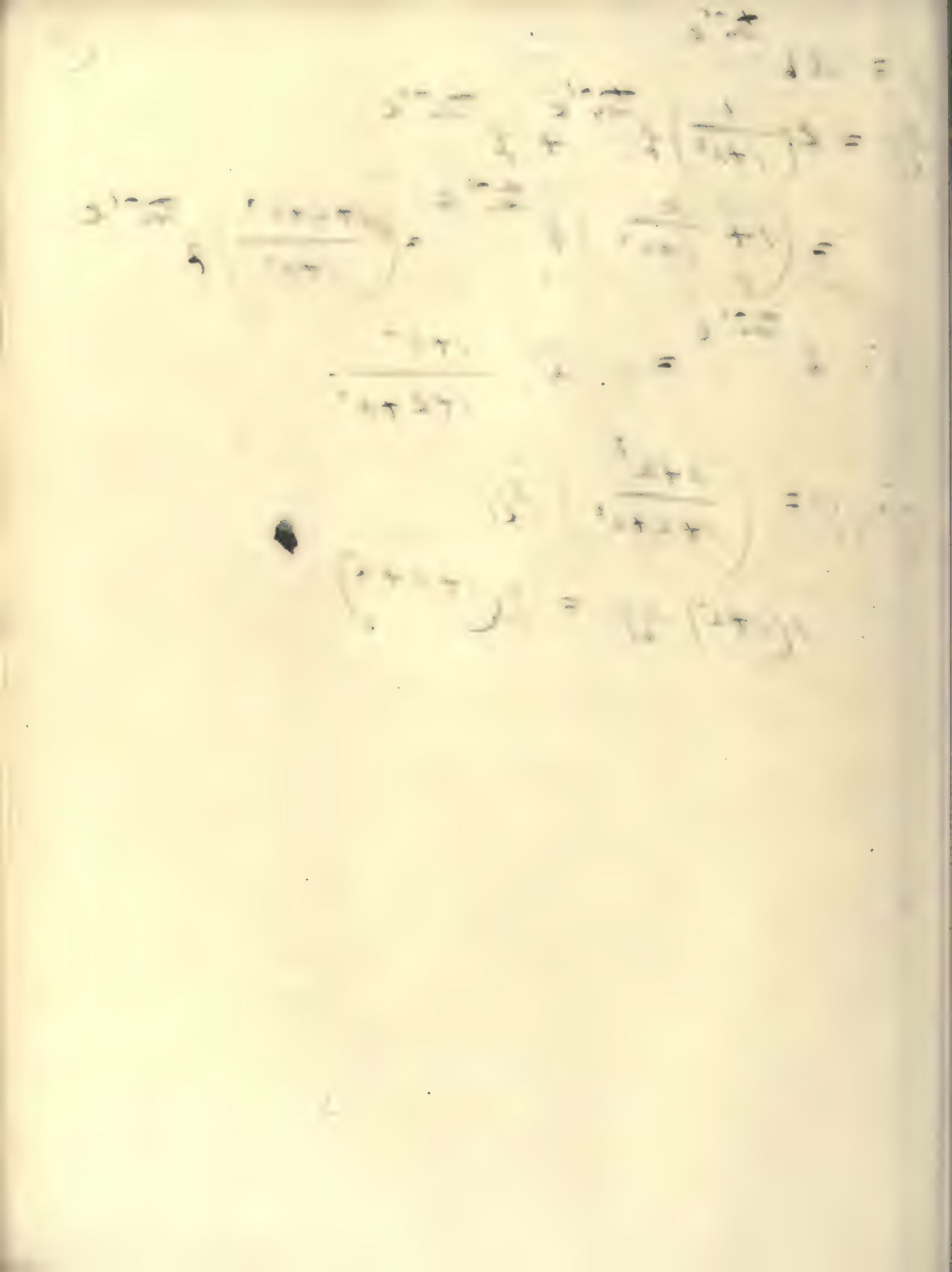
$\frac{dy}{dx} = x\left(\frac{1}{1+x^2}\right)a^{\tan^{-1}x} + a^{\tan^{-1}x}$.

$= \left(1 + \frac{x}{1+x^2}\right)a^{\tan^{-1}x} = \left(\frac{1+x+x^2}{1+x^2}\right)a^{\tan^{-1}x}$.

$\therefore a^{\tan^{-1}x} = \frac{dy}{dx}\,\frac{1+x^2}{1+x+x^2}$.

$\therefore y = \left(\frac{a+x^3}{1+x+x^2}\right)\frac{dy}{dx}$

$\therefore x(1+x^2)\frac{dy}{dx} = y(1+x+x^2)$



$$y = \frac{\xi^n + 1}{\xi^n - 1} \quad \therefore y(\xi^n - 1) = \xi^n + 1$$

$$\therefore \xi^n = \frac{y+1}{y-1}$$

$$\text{diff.} \quad \frac{(y-1)\,dy - (y+1)\,dy}{(y-1)^2} = n\xi^{n-1} = -\frac{2\,dy}{(y-1)^2}$$

$$\therefore dy \therefore \xi^n = -\frac{dy}{(y-1)^2}.$$

$$y = \frac{\xi^n + 1}{\xi^n - 1} \quad \therefore = -\frac{(y-1)^2 - dy}{dy + (y-1)^2} = \frac{dy - (y-1)^2}{dy + (y-1)^2}.$$

$$\therefore (y-1)\,dy + y(y-1)^2 + (y-1)^2 = 0$$

$$dy + y(y-1) + (y-1) = 0$$

$$dy + y^2 - 1 = 0.$$

(22)

$y = a\sin(x+b)$

$\frac{dy}{dx} = a\cos(x+b)$

$\frac{d^2y}{dx^2} = -a\sin(x+b) = -y \quad \therefore \frac{d^2y}{dx^2} + y = 0$

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(23)

$y = ae^x + be^{-x}$

$\frac{dy}{dx} = ae^x - be^{-x}$

$\frac{d^2y}{dx^2} = ae^x + be^{-x} = y \quad \therefore \frac{d^2y}{dx^2} + y = 0$

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(24)

$2y = ae^x + be^{-x} + c$

$2\frac{dy}{dx} = ae^x - be^{-x} \quad \therefore \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left\{a^2e^{2x} - 2ab + b^2e^{-2x}\right\}$

$2\frac{d^2y}{dx^2} = ae^x + be^{-x} \quad \therefore \left(\frac{d^2y}{dx^2}\right)^2 = \frac{1}{4}\left\{a^2e^{2x} + 2ab + b^2e^{-2x}\right\}$

$\therefore \left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{dy}{dx}\right)^2 + ab$

$\therefore \frac{dy}{dx} = \sqrt{ab + \left(\frac{dy}{dx}\right)^2}$.

(25)

$y = \log \sin(ax+b)$

$\frac{dy}{dx} = \frac{a\cos(ax+b)}{\sin(ax+b)} = a\tan(ax+b)$.

$\frac{d^2y}{dx^2} = a^2 \cdot \sec^2(ax+b) = a^2\{\tan^2(ax+b)+1\}$.

$= \left(\frac{dy}{dx}\right)^2 + a^2$

$\therefore \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + a^2$.

---

(27)

$ab^y = x^2 - b^2$

$\frac{dx}{dy}\frac{dy}{dx} = 1x \quad \therefore ab^y = \frac{1x}{\frac{dy}{dx}}$

$ab^y \cdot \frac{d^2y}{dx^2} + ab^y\left(\frac{dy}{dx}\right)^2 = 2$

$\therefore ab^y = \frac{2}{\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2} = \frac{1x}{\frac{dy}{dx}}$

$\therefore \frac{dy}{dx} = x\frac{d^2y}{dx^2} + a\left(\frac{dy}{dx}\right)^2$

$$y = mx \log_e \frac{max}{b(1-ax)}.$$

$$\frac{dy}{dx} = m \log_e \frac{max}{b(1-ax)} + \frac{m}{(1-ax)} = \frac{y}{x} + \frac{m}{1-ax}$$

$$\frac{d^2y}{dx^2} = \frac{x\frac{dy}{dx} - y}{x^2} + \frac{ma}{(1-ax)^2}$$

$$= \frac{y}{x^2} + \frac{m}{x(1-ax)} - \frac{y}{x^2} + \frac{ma}{(1-ax)^2}$$

$$= \frac{m}{x(1-ax)^2}$$

Now $\left(y - x\frac{dy}{dx}\right)^2 = \frac{m^2x^2}{(1-ax)^2}$

also $\frac{d^2y}{dx^2} \cdot mx^3 = \frac{m^2x^2}{(1-ax)^2}$

$\therefore mx^3 \cdot \frac{d^2y}{dx^2} = \left(y - x\frac{dy}{dx}\right)^2$

$y = \log e^{x} + e^{-x}$.

$\frac{dy}{dx} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$ (1) $\frac{d^2y}{dx^2} = \frac{(e^{x}+e^{-x})(e^{x}+e^{-x}) - (e^{x}-e^{-x})^2}{(e^{x}+e^{-x})^2}$ (2)

$= \frac{4}{(e^{x}+e^{-x})^2}$

$\therefore \left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2} y = \left(\frac{e^{x}+e^{-x}}{e^{x}+e^{-x}}\right)^2 = 1$

also from (1) $\frac{dy}{dx}(e^{x}+e^{-x}) = e^{x} - e^{-x}$.

$\log \frac{dy}{dx} + \log(e^{x}+e^{-x}) = \log(e^{x}-e^{-x})$

or $\frac{dy}{dx} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$

$\therefore \frac{dy}{dx}(e^{2x}+1) = e^{2x} - 1$

$e^{2x}\left(\frac{dy}{dx} - 1\right) = -1 - \frac{dy}{dx}$

$\therefore e^{2x} = \frac{1 + \frac{dy}{dx}}{1 - \frac{dy}{dx}}$

$\therefore x = \log \frac{1 + \frac{dy}{dx}}{1 - \frac{dy}{dx}}$

20)

$y = e^{\sec^{-1}x}$. $\quad dy = e^{\sec^{-1}x} \cdot \frac{1}{x\sqrt{x^2-1}} = \frac{y}{x\sqrt{x^2-1}}$

$\therefore d^2y = \left\{ \frac{1}{x^2(x^2-1)} dx - \frac{2x^2-1}{x^2(x^2-1)^{\frac{3}{2}}} \right\} e^{\sec^{-1}x}$.

$\sec^{-1}x = \log y$.

$\therefore \frac{1}{x\sqrt{x^2-1}} = \frac{dy}{y}$ $\quad \therefore dy = \frac{y}{x\sqrt{x^2-1}}$

$\therefore dy = \frac{y}{x\sqrt{x^2-1}}$

$d^2y = x\sqrt{x^2-1}\, dy - y.\left\{ \sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}} \right\}$

$x^2(x^2-1)$

$= \frac{dy}{x\sqrt{x^2-1}} - \frac{y(2x^2-1)}{x^2(x^2-1)^{\frac{3}{2}}}$

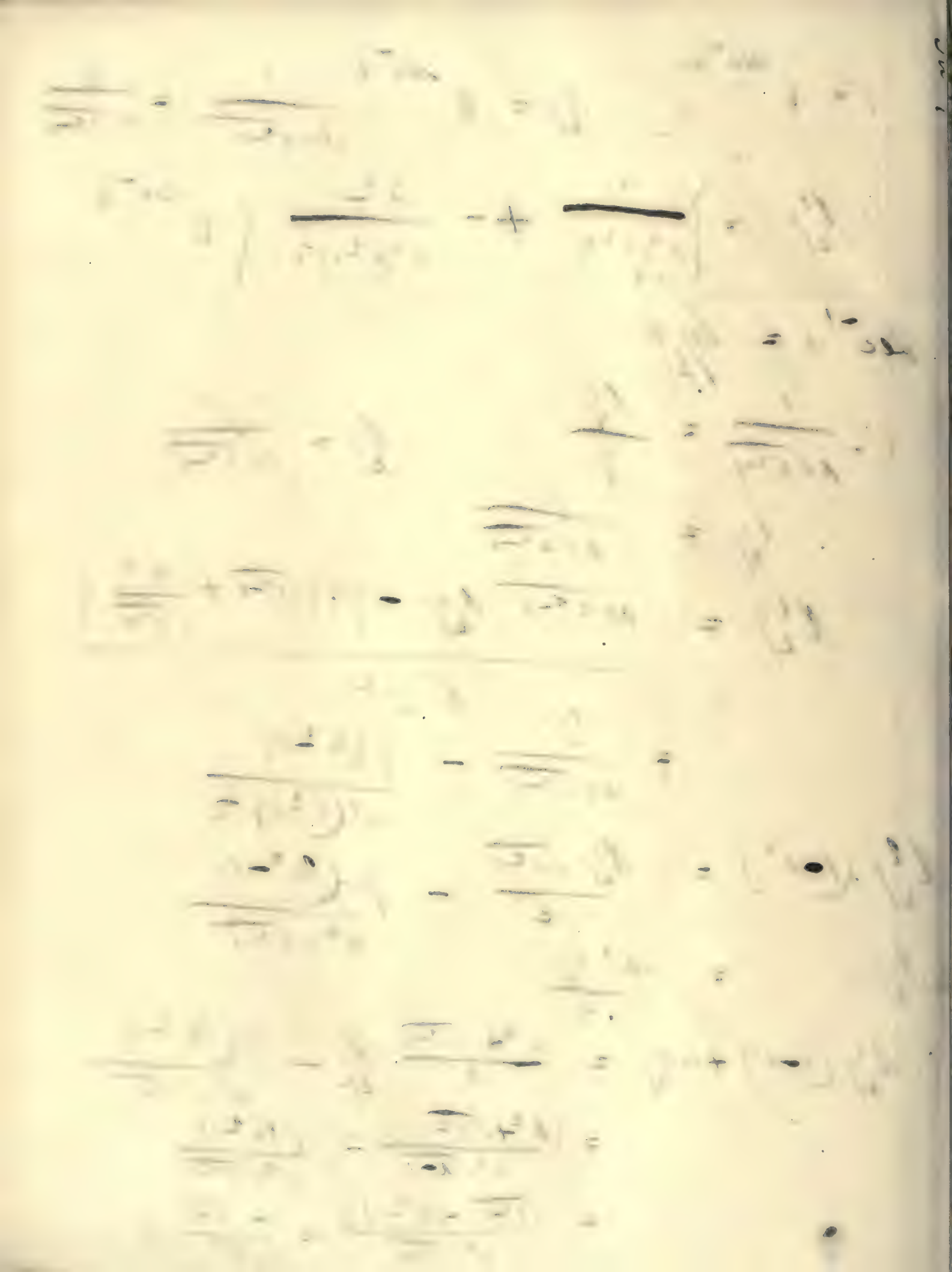
$d^2y.(1-x^2) = \frac{dy\sqrt{x^2-1}}{x} - \frac{y.(2x^2-1)}{x^2\sqrt{x^2-1}}$

$x\,dy = \frac{x^2\,dy}{x}$.

$\therefore d^2y(1-x^2) + x\,dy = \frac{x^2+\sqrt{x^2-1}}{x} \cdot dy - y\frac{(2x^2-1)}{x^2\sqrt{x^2-1}}$

$= \frac{(x^2+\sqrt{x^2-1})y}{x^2\sqrt{x^2-1}} - y\frac{(2x^2-1)}{x^2\sqrt{x^2-1}}$

$= \frac{\sqrt{x^2-1} - (x^2-1)y}{x^2\sqrt{x^2-1}} = \frac{1-\sqrt{x^2-1}}{x^2} \cdot y$



$y^3 - 2xy^2 + x^2y - a^3 = 0.$

Let $x = \frac{1}{z}$ $\therefore y^3 - \frac{2y^2}{z} + \frac{y}{z^2} - a^3 = 0$

or $z^2y^3 - 2y^2z + y - a^3z^2 = 0$ $\therefore z = 0$ gives $y = 0$.

Diff. and we have.

$2zy^3 + 3z^2y^2 d_z y - 4yz d_z y - 2y^2 + d_z y - 2az^3 = 0.$

$\therefore d_z y = \frac{2az^3 - 2zy^3}{3z^2y^2 - 4yz + 1} = 0$ if $z = 0$.

$z^2 d_z y + 2y^3 + 3z^2y^2 d_z y^2$

$6zy^2 d_z y + 6z^2y (d_z y)^2 - 4yz d_z^2 y - 4y d_z y - 4(d_z y)^2 - 4y d_z y + d_z^2 y - 2a^3 =$

But $d_z y = 0$

$\therefore d_z^2 y = \frac{2a^3 - 2y^3}{3z^2y^2 - 4yz + 1} = 2a^3$ if $z = 0$. $y = 0$.

$y^2 z d_z^2 y + 6y^2 d_z y^2 + 3z^2y^2 d_z^3 y - 4yz d_z^3 y + d_z^2 y = 0.$

$\therefore y = 0$ $\therefore d_z^3 y = \frac{0}{3z^2y^2 - 4yz + 1} = 0.$

$\therefore y = 0 + 0 \times z + \frac{2a^3 \cdot z^3}{12} + 0 \times \frac{z^3}{12.3.4} + \alpha$

$= \frac{a^3}{3a^2} + \alpha$

$my^3 - x^3y - mx^3 = 0.$

Let $x = \frac{1}{z}$. Then substitute.

$my^3 - \frac{y}{z^3} - \frac{m}{z^3} = 0 \therefore my^3z^3 - y - m = 0$

$\therefore$ if $z = 0$ $y = -m$.

$3my^3z^2 + 3my^2z^3\frac{dy}{dz} - \frac{dy}{dz} = 0$

$\therefore \frac{dy}{dz} = \frac{3my^3z^2}{1 - 3my^2z^3} = 0$ if $z = 0$.

$my^3z + 9my^2z^2\frac{dy}{dz} + 3my^2z^3\frac{d^2y}{dz^2}$

$+ 9my^2z^2\frac{dy}{dz} + 6myz^3\left(\frac{dy}{dz}\right)^2 - \frac{d^2y}{dz^2} = 0.$ But $\frac{dy}{dz} = 0$

$\therefore 6my^3z + 3my^2z^3\frac{d^2y}{dz^2} - \frac{d^2y}{dz^2} = 0$ or $\frac{d^2y}{dz^2} = \frac{6my^3z}{1 - 3my^2z^3} = 0$ if $z = 0$.

Differentiate again and substitute. Since $\frac{dy}{dz} = \frac{d^2y}{dz^2} = 0$.

$6my^3 + 3my^2z^3\frac{d^3y}{dz^3} - \frac{d^3y}{dz^3} = 0$

$\therefore \frac{d^3y}{dz^3} = \frac{6my^3}{3my^2z^3 - 1}$ if $z = 0$ and $y = -m$

This becomes. $\frac{-6m^4}{-3m\ 1} = -6m^4.$

$\therefore$ substitute

$y = -m + 0 \times \frac{z}{1} + 0 \times \frac{z^2}{1.2.} - \frac{6m^4}{1.2.3.}z^3 + \alpha.$

But $z = \frac{1}{x}$

$\therefore y = -m - \frac{m^4}{x^3} + \alpha.$

$u = \sin k \cos k$.

$\frac{du}{dk} = \cos k \cdot \cos k + \sin^2 k$.

$= \cos k \{ 1 - \cos k \} + (1 + \cos k)(1 - \cos k)$

$= 1 + 2\cos k \{ 1 - \cos k \} = 0$ if $\cos k = 1$ or $\cos k = -\frac{1}{2}$.

$\frac{d^2u}{dk^2} = (-2\sin k)(1 - \cos k) + (1 + 2\cos k)\cos k$.

$= (2\cos k - 2 + 1 + 2\cos k)\cos k$

$= (4\cos k - 1)\cos k = 0$ if $\cos k = 1$

$= -3\{\sqrt{1 - \frac{1}{4}} = -\frac{3}{2}\sqrt{3}$ if $k = -\frac{1}{2}$.

$\therefore u = 0$ a min if $\cos k = 1$. neither a max nor min.

$u = \frac{1}{2}\sqrt{3}\{1 + \frac{1}{2}\} = \frac{3}{4}\sqrt{3}$ a max. if $\cos k = -\frac{1}{2}$

or $k = \frac{2\pi}{3}$.

$u = \sin k \cos^2 k$.

$\frac{du}{dk} = \cos^3 k + 2 \cos k \sin^2 k \times -1$

$= \left(\cos^2 k - 2\sin^2 k\right) \cos k = 0$ if $k = \frac{\pi}{2}$ of $\cos k = 0$.

$= \left(1 - 3\sin^2 k\right) \cos k = 0$ if $\sin k = \frac{1}{\sqrt{3}}$.

$\frac{d^2u}{dk^2} = \left(3\sin^2 k - 1\right) \sin k + 6 \cos^2 k \sin k \times (-1)$

$= \left(3\sin^2 k - 1 - 6\cos^2 k\right) \sin k$

$\left(9\sin^2 k - 7\right) \sin k = 2$ if $k = \frac{\pi}{2}$ a minimum.

$= \frac{-4}{\sqrt{3}}$ a maximum.

$\therefore$ if $k = \frac{\pi}{2}$ $u = 0$ a minimum

if $k = \sin^{-1} \frac{1}{\sqrt{3}}$ we have

$u = \frac{1}{\sqrt{3}} \left\{ \frac{2}{3} = \frac{2}{3^{\frac{3}{2}}} \right.$ a maximum

$$u = \cos x \sin(a-x)$$

$$\frac{du}{dx} = -\{\sin x \sin(a-x) + \cos x \cos(a-x)\}.$$

$$= -\cos(a-2x) = 0 \text{ if } a-2x = \frac{\pi}{2} \text{ or } x = \frac{a}{2} - \frac{\pi}{4}$$

$$= 0 \text{ if } a-2x = -\frac{\pi}{2} \text{ or } x = \frac{a}{2} + \frac{\pi}{4}$$

$$\frac{d^2u}{dx^2} = -2\sin(a-2x) \text{ which is negative if } x = \frac{a}{2} - \frac{\pi}{4}$$

but positive if $x = \frac{a}{2} + \frac{\pi}{4}$.

$\therefore x = \frac{a}{2} - \frac{\pi}{4}$ gives $u = \cos\left(\frac{a}{2} - \frac{\pi}{4}\right)\sin\left(\frac{a}{2} + \frac{\pi}{4}\right)$

a maximum.

$x = \frac{a}{2} + \frac{\pi}{4}$ gives $u = \cos\left(\frac{a}{2} + \frac{\pi}{4}\right)\sin\left(\frac{a}{2} - \frac{\pi}{4}\right)$

a minimum

$u = e^{kx} \cos(x-a)$

$\frac{du}{dx} = e^{kx} \{ \cos(x-a) - k \sin(x-a) \}$

$= 0$ if $\cos(x-a) = k \sin(x-a)$ or $(x-a) = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ or $\frac{5\pi}{4}$

$= 0$ if $e^{kx} = 0$ or $x = -\infty$ $\quad \therefore x = \frac{\pi + 4a}{4}$.

$\frac{d^2u}{dx^2} = e^{kx} \{ \cos(x-a) - 2k \sin(x-a) - \cos(x-a) \}$.

$= -e^{kx} \cdot 2k \sin(x-a)$ which is negative if $(x-a) = \frac{\pi}{4}$.

$= 0$ if $e^{kx} = 0$.

which is positive if $(x-a) = \frac{5\pi}{4}$

$\therefore u = e^{k \frac{5\pi}{4}}$ is a maximum

$= e^{a + \frac{\pi}{4}} \cdot \frac{1}{\sqrt{2}}$.

also $u = e^{a + \frac{5\pi}{4}}$

$= -\frac{}{\sqrt{2}}$ is a minimum

$$u = \frac{e^{k}}{\sin(k-a)} = e^{k} \cdot \operatorname{cosec}(k-a).$$

$$\frac{du}{dk} = e^{k}\{\operatorname{cosec}(k-a) + \tan(k-x - \operatorname{cosec}(k-a)\}$$

$$= e^{k}\{1 - \cot\tan(k-a)\}\operatorname{cosec}(k-a)$$

$$= 0 \text{ if } \cot\tan(k-a) = 1 \text{ or } \operatorname{cosec}(k-a) = 0.$$

$$\text{or } k-a = \frac{\pi}{4} \therefore k = a + \frac{\pi}{4}$$

$$\frac{d^2u}{dk^2} = e^{a}\{1 - \cot(k-a) + \operatorname{cosec}^2(k-a) + -\cot(k-a) + \cot^2(k-a)\}\operatorname{cosec}(k-a)$$

$$= e^{a}\{1 - 1 + 2 - 1 + 1\}\sqrt{2}$$

$$= e^{a+\frac{\pi}{4}} \cdot 2\sqrt{2} \quad \text{a minimum if } (k-a) = \frac{\pi}{4}.$$

$u^3 - a^2x + x^3 = 0$.

$\therefore 3u^2 \frac{du}{dx} - a^2 + 3x^2 = 0 \quad (1)$

$$\frac{du}{dx} = \frac{a^2 - 3x^2}{3u^2}$$

Diff (1) and we have.

$3u^2 \frac{d^2u}{dx^2} + 6u\left(\frac{du}{dx}\right)^2 + 6x = 0$.

But for a max or min $\frac{du}{dx} = 0$

$\therefore 3u^2 \frac{d^2u}{dx^2} + 6x = 0$ or $\frac{d^2u}{dx^2} = \frac{-2x}{u^2}$.

Also since $\frac{du}{dx} = 0$ when $a^2 - 3x^2 = 0 \therefore x = \pm \frac{a}{\sqrt{3}}$.

But $u^3 = a^2x - x^3$

$$= a^2\left\{\frac{a}{\sqrt{3}}\right| - \frac{a^3}{3\sqrt{3}}\Big\} = \frac{a^3}{3\sqrt{3}}\left\{\sqrt{3} - 1\right\}.$$

$$\therefore u = \frac{a}{\sqrt{3}}\left\{3^{\frac{1}{2}} - 1\right\}^{\frac{1}{3}}$$

or $u^3 = a^2x - \frac{a}{\sqrt{3}} + \frac{a^3}{3\sqrt{3}} = \frac{a}{\sqrt{3}}(2)^{\frac{1}{3}}$

$$= -\frac{a^3}{3\sqrt{3}}\left\{3 - 1\right\} \therefore u = -\frac{a}{\sqrt{3}}\,2^{\frac{1}{3}}.$$

~~and both these values make $\frac{d^2u}{dx^2}$ negative~~

the former is a max. the latter a minimum.

$u^2 + 2uk^2 + 4k - 3 = 0$ (1)

$2u\,du + 2k^2\,du + 4uk + 4 = 0$. (2) $\therefore \frac{du}{dk} = -\frac{2(uk+1)}{u+k^2}$

$2u\left(\frac{du}{dk}\right)^2 + 2\left(\frac{du}{dk}\right)^2 + 2k^2\frac{du}{dk} + 4k\frac{du}{dk} + 4k\frac{du}{dk} + 4u = 0$

Before a max or min. we must have $\frac{du}{dk} = 0$

$\therefore 2u\,du^2 + 2k^2\,du^2 + 4u = 0$.

$\therefore du^2 = \frac{-4u}{2u + 2k^2} = -\frac{2u}{u+k^2}$.

Also when $\frac{du}{dk} = 0$ $uk + 1 = 0$

$\therefore u = -\frac{1}{k}$.

Substitute in (1)

$\frac{1}{k^2} - 2k + 4k - 3 = 0$.

$1 + 2k^3 - 3k^2 = 0$ $\therefore k = 1$ and $u = -1$.

$k = -\frac{1}{2}$ and $u = 2$.

$du^2 = \frac{-2u}{u+k^2}$ when $+ve$ if $k = 1$ and $u = -1$

and neg. if $k = -\frac{1}{2}$ $u = 2$

$\therefore k = 1$ and $u = -1$ gives both a max. & min.

$k = -\frac{1}{2}$ $u = 2$ — a max.

$u^4 - 4ux + x^4 + 2 = 0$. (1)

$4u^3 \frac{du}{dx} - 4x \frac{du}{dx} - 4u + 4x^3 = 0 \therefore \frac{du}{dx} = \frac{u - x^3}{u^3 - x}$

$u^3 \frac{d^2u}{dx^2} + 3u^2 \left(\frac{du}{dx}\right)^2 - 4x \frac{d^2u}{dx^2} - \frac{du}{dx} - \frac{du}{dx} + 3x^2 = 0$

But for a max or min $\frac{du}{dx} = 0$

$\therefore u^3 \frac{d^2u}{dx^2} + 3u^2 \frac{d^2u}{dx^2} + 3x^2 = 0$

$\therefore \frac{d^2u}{dx^2} = \frac{-3x^2}{u^3 + 3u^2}$

Now $\frac{du}{dx} = \frac{u - x^3}{u^3 - x} = 0 \quad \therefore \quad u = x^3$

Therefore from (1) $x^{12} - 4x^4 + x^4 + 2 = 0$

or $y^3 - 3y + 2 = 0$ when $y = x^4$.

$\therefore y = 1$ and $x = 1 \therefore u = 1$.

$y = -2$ and $x = \sqrt[4]{-2}$ which is impossible.

If $u = 1 \quad x = 1 \quad \frac{d^2u}{dx^2} = \frac{-3}{4}$ which is negative.

$$u = \frac{x^2 - a^2}{x^2} \tan \frac{\pi x}{2a}.$$

$$= 0 \times \infty \text{ if } x = a.$$

Ex. $u = \dfrac{\dfrac{x^2 - a^2}{x^2}}{\dfrac{1}{\tan \frac{\pi x}{2a}}} = \dfrac{\frac{x^2 - a^2}{x^2}}{\cot \frac{\pi x}{2a}} = \frac{0}{0}$ if $x = a$.

$$\frac{du}{dx} = \frac{\frac{2}{x} + \frac{2a^2}{x^3}}{-\frac{\pi}{2a} \operatorname{cosec}^2 \frac{\pi x}{2a}} = -\frac{\frac{2}{a}}{\frac{\pi}{2a}} \text{ if } x = a = -\frac{4}{\pi}.$$

32

$$u = \sec \frac{\pi x}{2} \log \frac{1}{x} = 0 \times \infty \text{ if } x = 1.$$

$$u = \frac{\log \frac{1}{x}}{\frac{1}{\sec \frac{\pi x}{2}}} = \frac{\log \frac{1}{x}}{\cos \frac{\pi x}{2}} = \frac{0}{0} \text{ if } x = 1$$

$$\frac{du}{dx} = + \frac{\frac{1}{x}}{\frac{\pi}{2} \sin \frac{\pi x}{2}} = \frac{2}{\pi} \text{ if } x = 1.$$

(33

$$\frac{u}{v} = \frac{\tan(a+x) - \tan(a-x)}{\tan^{-1}(a+x) - \tan^{-1}(a-x)} = \frac{0}{0} \text{ if } x = 0.$$

$$\frac{d_x u}{d_x v} = \frac{\sec^2(a+x) + \sec^2(a-x)}{\frac{1}{1+a^2+2ax+x^2} + \frac{1}{1+a^2-2ax+x^2}} = \frac{2\sec^2 a}{\frac{2}{1+a^2}} \text{ if } x = 0.$$

$$= \sec^2 a (1+a^2)$$

(34

$$u = \left(\sec \frac{\pi x}{2}\right)^2 \cdot \operatorname{vers} 2\pi x = \Delta \times 0 \text{ if } x = 1$$

$$\frac{u}{v} = \frac{\operatorname{vers} 2\pi x}{\frac{1}{\left(\sec \frac{\pi x}{2}\right)^2}} = \frac{\operatorname{vers} 2\pi x}{\cos^2 \frac{\pi x}{2}} = \frac{0}{0} \text{ if } x = 1$$

$$\frac{d_x u}{d_x v} = \frac{2\pi \sin 2\pi x}{\frac{\pi}{2} \cos \frac{\pi x}{2} \sin \frac{\pi x}{2}} = + \frac{2 \sin 2\pi x}{\cos \frac{\pi x}{2} \sin \frac{\pi x}{2}} = \frac{0}{0} \text{ if } x = 1$$

$$\frac{d_x^2 u}{d_x^2 v} = + \frac{4\pi \cos 2\pi x}{\frac{\pi}{2} \cos^2 \frac{\pi x}{2} - \frac{\pi}{2} \sin^2 \frac{\pi x}{2}} = \beta \text{ if } x = 1$$

$$\frac{u}{v} = \frac{\cos^{-1}(1-x)}{\sqrt{2x-x^2}} = \frac{0}{0} \text{ if } x=0.$$

$$\frac{du}{dx} = \frac{+1}{\sqrt{1-(1-2x+x^2)}} = \frac{1}{\sqrt{2x-x^2}}$$

$$\therefore v = (2x-x^2)^{\frac{1}{2}} = (2x)^{\frac{1}{2}} - \frac{1}{2}\cdot 2x^{\frac{3}{2}} \propto.$$

$$\therefore \frac{dv}{dx} = \frac{1}{2}\frac{2}{(2x)^{\frac{1}{2}}} - \propto - - = \frac{1}{(2x)^{\frac{1}{2}}} - \frac{3.2}{4}x^{\frac{1}{2}}$$

$$\therefore \frac{\frac{du}{dv}}{\frac{dv}{dx}} = \frac{\frac{1}{\sqrt{2x-x^2}}}{\frac{1}{\sqrt{2x}} - \frac{3}{2}x^{\frac{1}{2}} \propto -}$$

$$= \frac{\frac{1}{\sqrt{1-\frac{x}{2}}}}{\frac{1}{\sqrt{1}} - \frac{3}{2\sqrt{2}}x \propto.} = \frac{1}{1} = 1 \text{ if } x=0.$$

$$u = \frac{1-3x^2}{2x^2(1-x^2)} - \frac{\pi}{2x \operatorname{Lan} \pi x}$$

$$\frac{u}{v} = \frac{(1-3x^2)(2x \operatorname{Lan} \pi x) - 2\pi x^2(1-x^2)}{4x^3 \operatorname{Lan} \pi x \cdot (1-x^2)} = \frac{0}{0}.$$

$$\frac{du}{dv} = \frac{(1-3x^2)(2 \operatorname{Lan} \pi x + 2\pi x \operatorname{Lec}^2 \pi x) - 12x^2 \operatorname{Lan} \pi x - 4\pi x(1-x^2) + 4\pi x^3}{12x^2 \operatorname{Lan} \pi x(1-x^2) + 4\pi x^3 \frac{1}{\pi x}(1-x^2) - 8x^4 \operatorname{Lan} \pi x}$$

$$\frac{d^2u}{dv^2} = \frac{-6x\{2\pi x \operatorname{Lec}^2 \pi x) + (1-3x^2)4\pi \operatorname{Lec}^2 \pi x - 12x^2 \pi \cdot \operatorname{Lec}^2 \pi x + (8+12)\pi x^2}{-8\pi x^4 \operatorname{Lec}^2 \pi x - 8x^4 \operatorname{Lec}^2 \pi x}$$

Retaining only such terms as do not vanish when $x = 1$

Let $x = 1$ and we have

$$\frac{d^2u}{d^2v} = \frac{-12 \operatorname{Lec}^2 \pi - 8 \operatorname{Lec}^2 \pi - 12 \operatorname{Lec}^2 \pi + 20}{-16 \operatorname{Lec}^2 \pi}$$

$$= \frac{-32+20}{-16} = \frac{12}{16} = \frac{3}{4}$$

$$\frac{u}{v} = \frac{\log \tan x}{\log \tan 2x} = \frac{-\infty}{-\infty} \text{ when } x = 0.$$

$$\frac{u}{v} = \frac{\dfrac{1}{\log \tan 2x}}{\dfrac{1}{\log \tan x}} = \frac{-0}{-0} \text{ when } x = 0.$$

$$\frac{\dfrac{du}{dx}}{\dfrac{dv}{dx}} = \frac{\dfrac{-2\sec^2 2x}{\log^2 \tan 2x \times \tan 2x}}{-\dfrac{\sec^2 x}{\tan x (\log \tan x)^2}} = \frac{2\tan^3 x \cdot \log^2(\tan x)\sec^2 2x}{\tan^3 2x \cdot \log^2 \tan 2x \sec^2 x} = \frac{0}{0}.$$

$$= \frac{2\tan x}{\tan 2x} \text{ when } x = 0.$$

$$\frac{\dfrac{d^2u}{dx^2}}{\dfrac{d^2v}{dx^2}} = \frac{2\sec^2 x}{2\sec^2 2x} = \frac{1}{1} = 1 \text{ when } x = 0.$$

$$u = 2^k \cdot \tan \frac{a}{2^k} = \infty \times 0 \text{ if } x = 0.$$

$$u = \frac{\tan \frac{a}{2^k}}{\frac{1}{2^k}} = \frac{0}{0} = \frac{\tan \frac{a}{1 + \log_\xi 2 \cdot \frac{x}{1} + \frac{1}{2} \log_\xi 2 \, x^2 + \dots}}{\frac{1}{1 + \log_\xi 2 \frac{x}{1} + \frac{1}{2} \log_\xi 2 \, x^2 + \dots}}.$$

$$\frac{d}{dx} u' = \sec^2 \frac{a}{1 + \log_\xi 2 \cdot \frac{x}{1} + \frac{1}{2} \log_\xi 2 \frac{x^2}{1 \cdot 2} + \dots} \times a \frac{(\log_\xi 2 + \frac{2}{2_1} \log_\xi 2 \, x + \dots)}{\left(1 + \log_\xi 2 \frac{x}{1} + \frac{1}{2} \log_\xi 2 x^2\right)^2}$$

$$v' = \frac{\log_\xi 2 + \log_\xi 2 \, x + \dots}{1 + \log_\xi 2 \frac{x}{1} + \frac{1}{2} \log_\xi 2 \frac{x^2}{1} + \dots}$$

$$\frac{\frac{d}{dx} u'}{a v'} = \frac{\sec^2 \frac{a}{1 + \log_\xi 2 \frac{x}{1} + \dots} \times a \left\{ \frac{\log_\xi 2 + \log_\xi 2 \cdot x + \dots}{1 + \log_\xi 2 \frac{x}{1} + \frac{1}{2} \log_\xi 2 \, x^2 \dots} \right.}{\frac{\log_\xi 2 + x \log_\xi 2 + \dots}{1 + \log_\xi 2 \cdot \frac{x}{1} + \frac{1}{2} \log_\xi 2 \, x^2 + \dots}}$$

$$= a \sec^2 \frac{a}{1 + \log_\xi 2 \cdot \frac{x}{1} + \dots}.$$

$$= a \sec^2 0 \text{ if } x = \infty.$$

$$= a.$$

$$u = \frac{a^x \sin ax - b^x \sin bx}{c^x \sin cx - g^x \sin gx} = \frac{0}{0} \text{ if } x = 0.$$

$$u = \left(1 + \log_g a \cdot x + \log_g^2 a \cdot \frac{x^2}{1.2.} + \alpha\right) \sin ax - \left(1 + \log_g b \cdot x + \log_g^2 b \frac{x^2}{1.2.} + \right) \sin bx$$

$$= \left(1 + \log_g c \cdot x + \log_g^2 c \cdot \frac{x^2}{1.2} + \alpha\right) \sin cx - \left(1 + x + \frac{x^2}{1.2} + \alpha\right) \sin gx.$$

$$u = \left(1 + \log_g a \cdot x + \gamma_2\right) a \cos ax + \left(1 + \log_g a + \log_g^2 a \cdot x + \gamma_2\right) \sin ax$$

$$- \left(1 + \log_g a \cdot x + \alpha_2\right) b \cos bx - \left(1 + \log_g b + \log_g^2 b\, x + \gamma_2\right) \sin bx.$$

$$= a - b \text{ if } x = 0.$$

$$u = \left(1 + \log_g c \cdot x + \log_g^2 c \cdot \frac{x^2}{1.2.} \alpha_2\right) c \cos cx + \left(1 + \log_g c + \log^2 c \cdot x + \gamma_2\right) \sin cx$$

$$- \left(1 + x + \frac{x^2}{1.2} + \gamma_2\right) g \cos gx - \left\{1 + 1 + x + \alpha_2\right\} \sin gx.$$

$$= c - g \text{ if } x = 0.$$

$$\therefore \frac{du}{dv} = \frac{a - b}{c - g} \text{ if } x = 0.$$

$$u = \frac{1}{\log_b(1+x)} - \frac{1}{x} = \infty - \infty \text{ if } x = 0.$$

$$\frac{u}{v} = \frac{x - \log_b(1+x)}{x \log_b(1+x)} = \frac{0}{0}.$$

$$\frac{du}{dx} = 1 - \frac{1}{1+x} = \frac{1+x-1}{1+x} = \frac{x}{1+x}.$$

$$\frac{dv}{dx} = \log_b(1+x) + \frac{x}{1+x} = \frac{(1+x)\log_b(1+x) + x}{(1+x)}$$

$$\therefore \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{x}{(1+x)\log_b(1+x) + x} = \frac{0}{0} \text{ if } x = 0.$$

$$\frac{\frac{d^2u}{dx^2}}{\frac{d^2v}{dx^2}} = \frac{1}{\log_b(1+x) + 2} = \frac{1}{2} \text{ if } x = 0.$$

$$u = \frac{x-1}{2x^2} + \frac{1}{(e^{2x}-1)x} = \alpha + \beta \text{ of } x = 0.$$

$$\frac{u'}{v'} = \frac{(x-1)x.\{e^{2x}-1\} + 2x^2}{2x^3(e^{2x}-1)}.$$

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{1.2.} + \frac{(2x)^3}{1.2.3} + \alpha.$$

$$\therefore e^{2x} - 1 = 2x + 2x^2 + \alpha e.$$

$$\frac{u'}{v'} = \frac{(x^2-x)(2x+2x^2+\alpha) + 2x^2}{2x^3(2x+2x^2+\alpha.)}$$

$$= \frac{2x^3 - 2x^2 + 2x^4 - 2x^3 + \frac{4}{3}x^5 - \frac{4}{3}x^4 + \frac{2}{3}x^6 - \frac{2}{3}x^5 \alpha - + 2x^2}{4x^4 + 4x^5 + \alpha.}$$

$$= \frac{\frac{2}{3}x^4 + \frac{2}{3}x^5 \text{ —}}{4x^4 + 4x^5 \text{ — —}}.$$

$$\therefore \frac{d^4 u}{dx^4} = \frac{4.3.2.1.\frac{2}{3} + 5.4.3.2.\frac{2}{3}.x}{4.3.2.1.\times 4 + 5.4.3.2.\times 4x} = \frac{2}{12} = \frac{1}{6} \text{ of } x = 0.$$

$$u = \frac{1 - x + \log x}{1 - (2x - x^2)^{\frac{1}{2}}}.$$

Let $x = (a + h)$ then $\log x = \log(a + h)$

$$= hx - \frac{h^2 x^2}{2} + \&c.$$

also $(2x - x^2)^{\frac{1}{2}} = (a^2 - h^2)^{\frac{1}{2}} = \pm x \mp \frac{1}{2}\frac{h^2}{x} \pm \&c.$

$$\therefore u = \frac{1 - x + hx - \frac{h^2 x^2}{12} + \&c}{v = 1 \pm x \mp \frac{1}{2}\frac{h^2}{x} \pm \&c}$$

$$\frac{du}{dv} = \frac{-1 + h - hx^2 + \&c}{\pm 1 \mp \frac{1}{2}\frac{h^2}{x^2} + \&c.}$$

$$= \pm 1 \text{ if } h = 0 \quad x = 1$$

$x+h = (x+h)$

$\therefore \log x = \log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{1.2.x^2}$

$x+h = a$

$\therefore \log(a+h) = \log a + \frac{h}{a} - \frac{h^2}{1.2.a^2} -$

also $\sqrt{2ax-x^2} = \sqrt{a^2-h^2}$ if $x = a+h$

$= a - \frac{h^2}{1.a} + \&c.$

$u = \dfrac{a-(a+h)-a\log a + a\left(\log a + \frac{h}{a} - \frac{h^2}{1.2.a^2} + \&c\right)}{a-\left(a - \frac{h^2}{2a} + \&c\right)}$

$= \dfrac{-\frac{h^2}{2a}}{\frac{h^2}{2a}} = -1$

$$u = \frac{\sqrt{x^2-a^2} - \sqrt{2ax-x^2} + a}{\sqrt[3]{x^3-a^3} - \sqrt{x^2-a^2}}$$

$x+h = a+h \therefore \sqrt{x^2-a^2} = \sqrt{2ah+h^2} = \sqrt{2ah} + \frac{1}{2}\frac{h^2}{\sqrt{2ah}} + \&c.$

$-\sqrt{2ax-x^2} = -\sqrt{a^2-h^2} = -a - \frac{1}{2}\frac{h^2}{a} + \&c.$

$\sqrt[3]{x^3-a^3} = \sqrt[3]{3a^2h+3ah^2+h^3} = \sqrt[3]{3a^2h} + \frac{1}{3}\frac{3ah^2}{(3a^2h)^{\frac{2}{3}}} + \&c.$

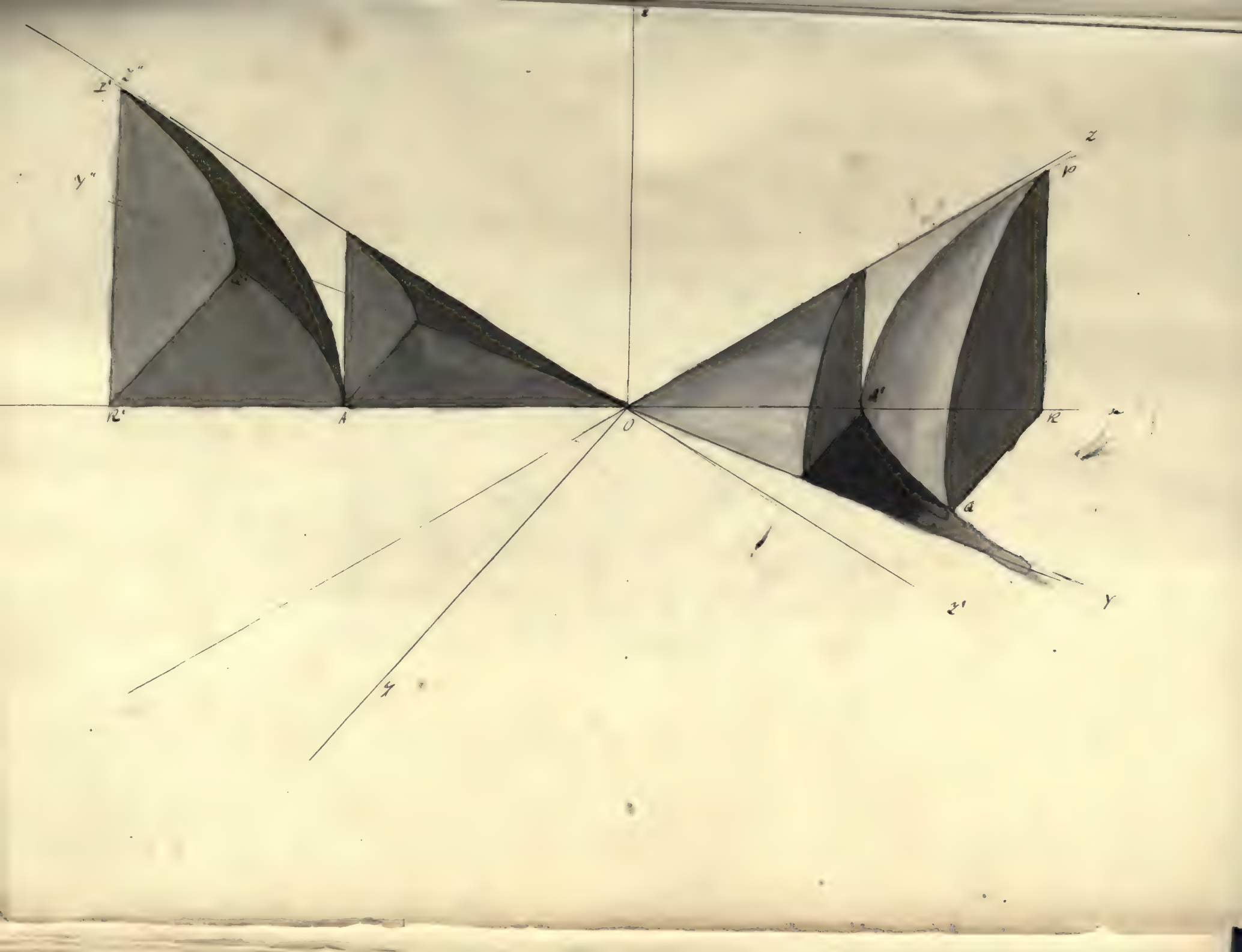
$\sqrt{x^2-a^2} = \sqrt{2ah} + \frac{1}{2}\frac{h^2}{\sqrt{2ah}} - \&c.$

$$u = \frac{\sqrt{2ah} + \frac{1}{2}\frac{h^{\frac{3}{2}}}{\sqrt{2a}} + \&c. - a + \frac{1}{2}\frac{h^2}{a} + \&c. + a}{\sqrt[3]{3a^2}.\sqrt[3]{h} + \frac{ah^{\frac{4}{3}}}{(3a^2)^{\frac{2}{3}}} + \&c. - \sqrt{2a}.h^{\frac{1}{2}} - \frac{1}{2}\frac{h^{\frac{3}{2}}}{\sqrt{2a}}}$$

$$= \frac{\sqrt{2a}.h^{\frac{3}{6}} + \&c. + \&c. -}{\sqrt[3]{3a^2}\,h^{\frac{2}{6}} + \&c. - - \sqrt{2a}.h^{\frac{3}{6}} - \&c.}$$

$$= \frac{\sqrt{2a}.h^{\frac{1}{6}} + \&c.}{\sqrt[3]{3a^2} + \&c. - - \sqrt{2a}.h^{\frac{1}{6}} + \&c.}$$

$$= \frac{0}{\sqrt[3]{3a^2}} \quad \text{ff. } h = 0 \quad = 0$$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MECHANICS.

TUESDAY, *January* 18, 1842,

Half-past Twelve o'clock to half-past Three.

Mr. TOZER and Mr. DRAKE, *Examiners.*

SECOND DIVISION.—(A)

1. DEFINE *weight, quantity of matter,* and *density.*

State the relations that subsist between them in bodies of the same magnitude.

2. What is a lever? Shew that a horizontal prism or cylinder of uniform density will produce the same effect by its weight as if it were collected at its middle point.

3. Two forces act at any angles on the arms of any lever. Enunciate and prove the condition necessary that they should balance each other.

State clearly what axioms, propositions, and assumptions you employ in this demonstration.

4. If the adjacent sides of a paralellogram represent component forces in magnitude and direction the diagonal will represent the resultant force in magnitude and direction.

5. Describe the several systems of pulleys, and determine for some one of them the ratio which the power must have to the weight to produce equilibrium.

6. Find the centre of gravity of two heavy points, and shew that the pressure at the centre of gravity is equal to the sum of the weights in all positions.

7. When a body is placed on a horizontal plane it will stand or fall, as the vertical through its centre of gravity falls within or without its base.

Hence shew why CÆTERIS PARIBUS *a body is more easily overturned as its base is diminished in size.*

8. The pressure upon any particle of a fluid of uniform density is proportional to its depth below the surface of the fluid.

9. If a vessel, the bottom of which is horizontal, and whose sides are vertical be filled with fluid, the pressure on the bottom will be equal to the weight of the fluid.

What will be the pressure on the bottom when the sides are not vertical?

10. When a body is immersed in a fluid, the weight lost is to the whole weight of the body as the specific gravity of the fluid to the specific gravity of the body.

JUVENAL.

FRIDAY, *January* 14, 1842,

Half-past Twelve o'clock to half-past Three.

SECOND DIVISION.—(A)

TRANSLATE: *Sat.* x., 90—109.

Visne salutari sicut Sejanus? habere
to
Ad sua qui domitos deduxit flagra Quirites?

Explain the terms *summas curules, pila, cohortes, gregios equites, castra domestica, prætextam,* Explain fully the historical allusions, giving the dates of the different events.

Translate: *Sat.* x., 329—345.

Elige quidnam
Suadendum esse putes, cui nubere Cæsaris uxor
to
Prœbenda est gladio pulchra hæc et candida cervix.

Who was the object of Messalina's passion alluded to in the present passage? What part of the marriage ceremonies is, strictly speaking, indicated by *nubere?* How do you explain the words *ritu antiquo?* Explain the Roman method of computing money.

Translate: *Sat.* XIII., 1—25.

Exemplo quodcunque malo committitur, ipsi
to
Quæsitum, et partos gladio vel pyxide numos?

Derive *Prætor.* Explain the words *fallaci urna.* Determine the date of this Satire from the words *qui jam post terga reliquit Sexaginta annos, &c.* &c.

Translate: *Sat.* XIV., 120—140.

Servorum ventres modio castigat iniquo,
to
In ventres abeant, ut credas falsibus actum.

Explain *aliquis de ponte.*

FRIDAY, *Jan.* 14, 1842,

Half-past Twelve o'clock to half-past Three.

(B)

Translate: *Sat.* x., 147—173.

Expende Hannibalem: quot libras in duce summo
to
Quantula sint hominum corpuscula.

Give an account of the life of Hannibal, with the dates of the siege of Saguntum, the battles of the Trebia, Thrasymene, Cannæ, and Zama, and of the commencement and close of the three Punic wars. What was the Subura? Where was Pella? When and where did Alexander die?

Translate: *Sat.* XIII., 174—198.

Nullane perjuri capitis, fraudisque nefandæ
to
Nocte dieque suum gestare in pectore testem.

To what sect of philosophy did Chrysippus belong? Who is meant by *dulci senex vicinus Hymetto?* Give an account of the particulars of his death, with its date.

Translate: *Sat.* XIV., 248—280.

Nota mathematicis genesis tua: sed grave
to
Audiet Hercule stridentem gurgite solem.

Explain the term *municipes Jovis.*

TUESDAY, *January* 11, 1842,

One o'clock to Four.

Mr. GREGORY, *Mod.*—Mr. THURTELL, *Ex.*

1. THE sum of the faces and solid angles of any polyhedron exceeds the number of edges by two.

2. Prove the theorem

$$f(e^x) = f(1+\Delta).1 + f(1+\Delta)0\frac{x}{1} + f(1+\Delta)0^2\frac{x^2}{1.2} + \&c.;$$

and apply it to demonstrate the following expression for the numbers of Bernoulli,

$$B_{2n-1} = (-)^{n-1}\left\{\frac{0^{2n}}{1} - \frac{\Delta 0^{2n}}{2} + \frac{\Delta^2 0^{2n}}{3} - \&c. + \frac{\Delta^{2n}0^{2n}}{2n+1}\right\}.$$

3. Define an umbilicus; and find the positions of the umbilici in the surface of which the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c},$$

What is the nature of the points in a surface at which,

$$\frac{dz}{dx} = \frac{0}{0}, \qquad \frac{dz}{dy} = \frac{0}{0}?$$

4. A tube open at both ends gives, when the air it contains is put in vibration, the same note as a tube of half the length which is closed at one end.

5. If a rigid body be struck by any couple, the axis of the couple can never be the corresponding instantaneous axis of rotation, unless it coincide with a principal axis of the body. Shew also that, when this condition is fulfilled, the angular velocity round the principal axis is equal to the moment of the couple divided by the moment of inertia round that axis.

6. When an electrised body is surrounded by a non-conducting medium, the pressure sustained by that medium is proportional to the square of the electrical accumulation, at each point of the surface of the body.

Describe the construction of the Leyden jar and the Proof Plane. To what purpose is the latter instrument applied?

7. Explain the difference between the periodic and secular variations in a planet's motion.

If two planets, of which the masses are m, m', and the semi-major axes of the orbits a, a', mutually disturb each other, the ratio of the coefficient of any long inequality of m to the corresponding coefficient for m' is equal to $-\frac{m'\sqrt{a'}}{m\sqrt{a}}$.

8. Assuming that the velocity of propagation of a plane wave in a medium is given by the equation

$$\frac{l^2}{a^2 - v^2} + \frac{m^2}{b^2 - v^2} + \frac{n^2}{c^2 - v^2} = 0,$$

explain how the equation to the wave-surface is investigated; and point out the peculiarities in that surface which indicate the existence of cylindrical and conical refraction, when pencils of light pass in certain directions through a biaxal crystal.

9. Assuming that, in the expression for the radius of a nearly spherical stratum of the earth, considered as a revolving heterogeneous mass originally fluid, all Laplace's coefficents vanish except that of the second order; prove that the strata of equal density are all concentric spheroids of revolution, whose axes coincide with that about which the earth revolves.

State the other results of the hypothesis of the Earth's original fluidity, the subsidiary assumptions made in obtaining them, and the evidence for the truth of the theory arising from the comparison of its results with observation.

MONDAY, *January* 10, 1842,

Nine o'clock to half-past Eleven.

Mr. GREGORY, *Mod.*—Mr. POTTER, *Ex.*

1. SHEW that the number of combinations of n letters taken r together is equal to that of the combinations of n letters taken $n-r$ together.

What is the number of terms in the expansion of $\{a_1 + a_2 + \&c. + a_n\}^m$?

2. When a body is placed on a horizontal plane, it will stand or fall, according as the vertical line, drawn from its centre of gravity, falls within or without the base.

3. If two imperfectly elastic balls impinge obliquely, determine their motions after impact.

If the mass of one of them be to that of the other as their common elasticity to unity, shew that the motion of the first after impact will be parallel to the plane touching the spheres at the point of contact, if the motion of the other ball were so previously.

4. If two quantities substituted for the unknown quantity in an equation give results with different signs, an odd number of possible roots lies between them; but if they give the results with the same signs, an even number or none at all lies between them.

5. A cord is stretched round a given arc of a cylinder; having given the coefficient of friction between the cord and the cylinder, find the force which, acting at one extremity of the cord will balance a force P at the other, P being on the point of preponderating.

6. Find the condition of achromastism in a double object-glass: point out the kinds of lenses of which it must be composed, and which of the lenses must have the lower dispersive power.

Trace an achromatised pencil through such an object-glass.

7. Find the equations to a tangent line at any point of a curve of double curvature: find also the equation to the locus of all tangent lines which can be drawn to a surface at one point.

8. If any forces act on a rigid body, and the resultant force be parallel to the axis of the resultant couple, the moment of the couple is a minimum.

9. Shew how to distinguish a singular solution from a particular integral of a given differential equation.

Is $y = x$ a singular solution or a particular integral of

$$(x+a)\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = a?$$

10. Explain the object and advantages of rifling the barrel of a gun.

11. State and prove the principle of Virtual Velocities; apply it to find the condition of equilibrium when three forces act on a point.

12. Shew generally how to transform one double integral into another with different variables, having given the relations between the variables.

If $x = u\sin\alpha + v\cos\alpha$ and $y = u\cos\alpha - v\sin\alpha$, prove that

$$\iint f(x,y)\frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \iint f_1(u,v)\frac{du\,dv}{\sqrt{1-u^2-v^2}}.$$

11. Prove that the orthogonal trajectory of a series of ellipses described round the same foci is a confocal hyperbola.

12. If a transparent plate lie between two media of unequal densities, each less than that of the plate, and if a ray, passing from the denser of the external media into the plate, fall at the critical angle on the third medium, the angle of incidence at the first surface of the plate will be the critical angle from the first on the third medium.

13. A thin surface in the form of a paraboloid of revolution is just immersed in a fluid, first with its base, next with its vertex downwards by means of two weights W, W': shew that if h be the height of a column of fluid whose weight is equal to the atmospheric pressure, $h = a\left(\frac{W}{W'}\right)^{\frac{1}{2}} \frac{W}{W' - W}$, a being the height of the paraboloid.

14. If $F\{x, y, z, a, b, c\} = \varphi\{u, v, w, a, b, c\}$, where F is homogeneous of the nth degree in x, y, z, and

$$u = \frac{dF}{dx}, \quad v = \frac{dF}{dy}, \quad w = \frac{dF}{dz},$$

prove that $x = (n-1)\frac{d\varphi}{du}$, $y = (n-1)\frac{d\varphi}{dv}$, $z = (n-1)\frac{d\varphi}{dw}$.

15. If a normal to the surface, of which the equation is

$$x \cos nz - y \sin nz = 0,$$

move along any one of its generating lines, it will trace out a hyperbolic paraboloid.

16. A quadrilateral formed of four unequal beams jointed together at their extremities is compressed by a given force in the direction of one diagonal: find the force in the direction of the other diagonal which will resist the compression.

17. A homogeneous rectangular beam turns freely round a horizontal axis passing through the middle point of one end and parallel to a terminal edge, while the other end dips into a fluid of greater specific gravity: find the position of equilibrium.

18. If $\varphi(a-x)$ be an algebraical function which vanishes when $x = c$, prove that

$$\int_0^c dx \varphi(a-x) \cdot x^{n-1} = \Gamma(n)\left(\frac{d}{da}\right)^{-n} \varphi(a).$$

Evaluate $\int_0^\infty \frac{(x^2+a^2)dx}{x^4+b^2x^2+b^4}$ and $\int_0^{\frac{1}{2}\pi} dx \cos(a \tan x)$.

Solve the functional equation

$$\varphi(x) + \varphi\{x + \varphi(x)\} = 0.$$

19. If p, $p_2, \cdots p_{2n-1}, p_{2n}$ be the perpendiculars drawn from any point in the circumference of a circle on the sides of a regular circumscribing polygon of $2n$ sides, shew that

$$p_1 \cdot p_3 \cdots p_{2n-1} + p_2 \cdot p_4 \cdots p_{2n} = \frac{a^n}{2^{n-2}},$$

where a is the radius of the circle.

20. If four similar surfaces of the second order intersect each other, the planes of their intersections two and two all pass through one point.

21. An inelastic cylinder rolls down a rough inclined plane, and strikes against another plane equally

FRIDAY, *January* 7, 1842,

One o'clock to Four.

Mr. GREGORY, *Moderator.*

1. IN a triangle ABC, if E, F be the points where perpendiculars from the opposite angles meet the sides AC, AB, shew that

$$BC^2 = AB \cdot BF + AC \cdot CE.$$

2. Prove that, if $\frac{p}{q}, \frac{p_1}{q_1}, \frac{p_2}{q_2}$ be three consecutive convergents of a continued fraction, $\frac{p_1}{q_1} = \frac{p_2 - p}{q_2 - q}$.

3. If a, b, c be the sides, A, B, C the angles of a triangle,

$$a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C:$$

R being the radius of the circumscribing circle.

4. If $C_r(n)$ be the number of combinations, and $P_r(n)$ the number of homogeneous products of r dimensions which can be made with n letters, shew that, so long as $n - m < r$,

$$P_r(m) - C_1(n) P_{r-1}(m) + C_2(n) P_{r-2}(m) - \&c. \pm C_r(n) = 0.$$

5. If lines drawn through any point in an ellipse to the extremities of any diameter meet its conjugate (CD) in points M, N, prove that

$$CM \cdot CN = CD^2.$$

6. A fixed circle is cut by a series of circles, all of which pass through two given points: shew that the lines which join the points of intersection of the fixed circle with each circle of the series all converge to one point.

7. A ball of given weight and radius is hung by a string of given length from a fixed point, to which also is attached another given weight by a string so long that the weight hangs below the ball: find the angle which the string, to which the ball is attached, makes with the vertical.

8. Two perfectly elastic balls are dropped from two points not in the same vertical line, and strike against a perfectly elastic horizontal plane: shew that their centre of gravity will never re-ascend to its original height, unless the initial heights of the balls be in the ratio of two square numbers.

9. If two straight lines at right angles to each other on an inclined stratum make angles α, β with the horizontal plane, the dip (θ) of the stratum is given by the equation $\cos\theta = \{\cos(\alpha+\beta)\cos(\alpha-\beta)\}^{\frac{1}{2}}$.

10. Two ships are sailing uniformly with velocities u, v along lines inclined at an angle θ; shew that if a, b be their distances at one time from the point of intersection of the courses, the least distance of the ships is equal to

$$\frac{(av - bu)\sin\theta}{(u^2 + v^2 - 2uv\cos\theta)^{\frac{1}{2}}}.$$

THURSDAY, *January* 6, 1842,

One o'clock to Four.

Mr. GASKIN, *Moderator.*

1. IF θ be the angle between the diagonals of a parallelogram whose sides a, b are inclined at angle α to each other, then $\tan\theta = \dfrac{2ab\sin\alpha}{a^2-b^2}$.

2. If $\cos^2\theta = \dfrac{\cos\alpha}{\cos\beta}$, $\cos^2\theta' = \dfrac{\cos\alpha'}{\cos\beta}$, and $\dfrac{\tan\theta}{\tan\theta'} = \dfrac{\tan\alpha}{\tan\alpha'}$; shew that $\tan\dfrac{\alpha}{2}\tan\dfrac{\alpha'}{2} = \tan\dfrac{\beta}{2}$.

3. From the extremities of any chord PQ of a conic section, draw the tangents PT, QT to meet in T; then if α be the angle which the chord PQ makes with its corresponding diameter, $\cot TPQ - \cot TQP = 2\cot\alpha$.

4. Shew that the length of the longer normal, drawn from a point in the minor axis of an ellipse at a distance c from its centre, and intercepted between that point and the curve $= \sqrt{a^2 + \dfrac{c^4}{e^2}}$.

5. If $x = ye^y$, shew that

$$\sin(\alpha+y) = \sin\alpha + a_1\frac{x}{1} + a_2\frac{x^2}{1.2} + \ldots + a_n\frac{x^n}{1.2\ldots n} + \&c.$$

where

$a_1 = \cos\alpha$; and $a_n = (-1)^{n-1}(1+n^2)^{\frac{n-1}{2}}\cos\{\alpha + (n-1)\cot^{-1}n\}$.

6. The velocities at the extremities of any chord of the parabola described by a body projected obliquely and acted on by gravity, when resolved in a direction perpendicular to the chord, are equal.

7. An elastic string passes over a tack, and is attached to the extremities of an uniform beam; supposing the increase of length of the string when stretched by the weight of the beam equal to twice the length of the beam, determine the position of equilibrium.

8. The sights of a gun are set so that the ball may strike a given object; shew that when the sights are directed to any other object in the same vertical line, the ball will also strike it.

9. If α, β be the distances of any two geometrical foci Q, q from the surface of a spherical refracting medium, and Q', q' any other two geometrical foci, shew that

$$\frac{\beta}{\alpha}\cdot\frac{1}{qq'} - \mu\frac{\alpha}{\beta}\cdot\frac{1}{QQ'} = \frac{\mu-1}{r}.$$

10. If a line be drawn from the focus of an ellipse whose eccentricity $= e$, so as to make a given angle α with the tangent; shew that the locus of its intersection with the tangent will be a circle which touches or falls entirely without the ellipse according as

$$\alpha > \text{or} < \cos^{-1}e.$$

that the portion of the vertical pressure supported by each is indeterminate.

9. Assuming the general relation between the pressure and the velocity in fluid motion, when the velocity at any point is independent of the time, find the approximate velocity with which an incompressible fluid, acted on by gravity, issues through a very small orifice in the vessel that contains it.

Ex. Find the approximate time of emptying half the fluid from a full conical vessel which has its axis vertical, and a small orifice at the vertex.

10. Shew that a ray of light enters a denser from a rarer medium at every angle of incidence. Find the critical angle when a ray in a denser medium falls on a rarer medium.

11. Find the attraction of a spherical shell on an internal and on an external particle, and hence the attraction of a solid sphere on a point within it, the force of attraction varying inversely as the square of the distance.

12. Define the centre of spontaneous rotation of a rigid body, and find its position in an uniform rod when struck at any point.

13. Find through what portion of its orbit the motion of a planet appears retrograde, supposing the Earth and planet to describe circles about the Sun. Shew that if the Earth were at rest the motion of a superior planet would never appear retrograde.

14. What causes may be presumed to affect the relative positions of two fixed stars, which are apparently near together? Shew how the effects of these different causes may be distinguished from each other.

15. Calculate the relative maxima and minima intensities in the bright and dark rings in Newton's experiment, seen by transmission, when $\mu = 1.5$, from the formula

$$\text{Intensity} = \frac{a^2 (1-e^2)^2}{(1-e^2)^2 + 4e^2 \sin^2 \frac{\pi V}{\lambda}},$$

Fresnel's expression giving $\left(\frac{\mu-1}{\mu+1}\right)^2$ as the intensity of the light reflected perpendicularly at the surface of a medium, when unity is the intensity of the incident light.

THURSDAY, *January* 13, 1842,

Half past Twelve o'clock to half-past Three.

SECOND DIVISION.—(B)

Mr. J. FENDALL, *Examiner.*

I. TRANSLATE:

Σκληροτράχηλοι, καὶ ἀπερίτμητοι τῇ καρδίᾳ καὶ τοῖς ὠσὶν, κ. τ. λ. to τὴν ἁμαρτίαν ταύτην. καὶ τοῦτο εἰπὼν ἐκοιμήθη.

Give the derivations of σκληροτράχηλοι, ἀπερίτμητοι, διεπρίοντο, ὁμοθυμαδὸν, ἐλιθοβόλουν. Explain the meaning of the expression εἰς διαταγὰς ἀγγέλων.

II. Translate:

Συνήχθησαν δὲ οἱ ἀπόστολοι καὶ οἱ πρεσβύτεροι, κ. τ. λ. to Ἰούδαν τὸν ἐπικαλούμενον Βαρσαβᾶν, καὶ Σίλαν, ἄνδρας ἡγουμένους ἐν τοῖς ἀδελφοῖς.

To what circumstance does Peter refer in the passage Ἄνδρες ἀδελφοὶ, &c. to πιστεῦσαι? Whose son was James and what office did he hold in the Church? What gave rise to this council? Explain the passage Μωϋσῆς γὰρ ἐκ γενεῶν, &c. to ἀναγινωσκόμενος. Where was Antioch? and what circumstances connected with the early Church occurred there? From whence is the quotation "Μετὰ ταῦτα ἀναστρέψω.....ὁ ποιῶν ταῦτα πάντα"?

III. Translate:

Ὡς δὲ ἔμελλον αἱ ἑπτὰ ἡμέραι συντελεῖσθαι, κ. τ. λ. to προσεφώνησε τῇ Ἑβραΐδι διαλέκτῳ λέγων.

What proportion of country was comprehended under the term "Asia"? What were the geographical positions of Ephesus and Tarsus? When was the temple here mentioned built, and when destroyed? In what consisted the crime imputed to St. Paul in the passage ἔτι τε καὶ Ἕλληνας εἰσήγαγεν εἰς τὸ ἱερὸν, &c.? Explain the passage ἐκέλευσε δεθῆναι ἁλύσεσι δυσί, and οὐκ ἄρα σὺ εἶ ὁ Αἰγύπτιος, &c. What was the office of χιλιάρχος?

EUCLID.

FRIDAY, *January* 14, 1842,

Nine o'clock to Twelve.

Mr. TOZER and Mr. ANSTED, *Examiners.*

FIRST DIVISION.—(A)

1. THE angles at the base of an isosceles triangle are equal, and if the equal sides be produced the angles on the other side of the base are equal.

2. To bisect a given finite straight line: that is, to divide it into two equal parts.

3. If from a point within a triangle two straight lines be drawn, one to each end of one side of the triangle, these two lines shall be less than the other two sides of the triangle, but shall contain a greater angle.

SATURDAY, *Jan.* 8, 1842,

One o'clock to half-past Four.

Mr. THURTELL, *Mod.*—Mr. POTTER, *Ex.*

1. EXPRESS 2ft. $10\frac{1}{4}$in. linear measure, and 5ft. $79\frac{1}{6}$in. square measure in the duodecimal scale of notation as feet and duodecimals of a foot; also, the latter quantity being the area of a rectangle one of whose sides is the former, find its other side by dividing in the duodecimal scale.

2. Find the sum of a converging geometrical series continued *ad infinitum*; and find the difference between the sum of an infinite number of terms and that of n terms, when the common ratio is a negative proper fraction.

3. Investigate a series for calculating the value of π, and find the value to 5 places of decimals.

4. If x and y be rectangular co-ordinates, what is represented by the equation

$$ax^2 + bxy + cy^2 = 0?$$

5. Find the angle between the tangent and radius vector of a curve referred to polar co-ordinates. Determine also whether the curve does, or does not, pass between the tangent and radius vector.

6. Explain what is meant by a diametral surface, a diametral plane, conjugate diametral planes, and principal diametral planes of surfaces. Shew that in surfaces of the second order the diametral surfaces are planes.

7. Ingrate the equations $(1 + x^2)\frac{dy}{dx} + xy + 1 = 0$,

$$y^3\frac{d^2y}{dx^2} = a^2.$$

Find the sum of any number of terms of the recurring series

$$0 + 4 - 16 + 48 - \&c.$$

8. Find the conditions of equilibrium when any forces act on a rigid body in one plane.

Ex. A door rests on two hinges in a vertical line; determine the lateral pressure on the hinges, and shew

MECHANICS.

Monday, *Jan.* 17, 1842.

Nine o'clock to Twelve.

Mr. DRAKE, *Examiner.*

First Division.—(A.)

1. State clearly the difference between the *density* of a body and its *weight*. What is meant by *Specific Gravity?* How does its definition differ from that of *density?*

2. If two weights acting perpendicularly on a straight lever on opposite sides of the fulcrum balance each other, they are inversely as their distances from the fulcrum, and the pressure on the falcrum is equal to their sum.

3. If two weights balance each other on a straight lever when it is horizontal, they will balance each other in every position of the lever.

Explain why a common scale beam, the arms of which are equally loaded, rests only in a horizontal position.

4. Define what is meant by Component and Resultant forces.

If three forces represented in magnitude and direction by the sides of a triangle act on a point they will keep it at rest.

5. Describe the several *systems of pulleys*, and determine for some one of them the ratio which the power must have to the weight to produce equilibrium.

6. If P and W balance each other on the wheel and axle and the machine be put in motion, then (assuming that the arcs which subtend equal angles at the centres of two circles are as the radii of the circles) $P : W :: W$'s velocity : P's velocity.

7. Define the *centre of gravity* of a system of bodies. Shew how to find its position when the system consists of any number of heavy points.

8. Define "*Fluid*" and distinguish between elastic and non-elastic fluids.

Shew that the surface of every fluid at rest is horizontal.

9. If a body floats on a fluid it displaces as much of the fluid as is equal in weight to the weight of the body, and it presses downwards and is pressed upwards with a force equal to the weight of the fluid displaced.

What pressure does the body sustain when it sinks in the fluid.

10. Describe the common *hydrometer*, and shew how to compare the specific gravity of two fluids by means of it.

11. Describe the *forcing pump* and its operation.

What is the greatest height to which water can be raised by the common pump, and why?

12. Having given the number of degrees on the *centigrade* thermometer, find the corresponding number on Fahrenheit's scale.

Ex. What degree of Fahrenheit corresponds to 45° *C?*

ACTS OF THE APOSTLES.

WEDNESDAY, *Jan.* 12, 1842.

Nine o'clock to Twelve.

Mr. ATKINSON, *Examiner.*

FIRST DIVISION.—(A.)

1. TRANSLATE, *Ch.* v., *v.* 33—42.

Οἱ δὲ ἀκούσαντες διεπρίοντο, κ. τ. λ. to *καὶ εὐαγγελιζόμενοι Ἰησοῦν τὸν Χριστόν.*

Mention what is known of the date and circumstances of the insurrection of Judas. Is the *ἀπογραφὴ* here spoken of the same as that recorded Luc. ii. 2.? Quote the passage in which the author of the Acts is supposed to refer to his Gospel. Give a succinct account of his connexion with St. Paul.

2. Translate: *Ch.* XII. *v.* 1—9.

Κατ' ἐκεῖνον δὲ τὸν καιρὸν ἐπέβαλεν Ἡρῴδης, κ. τ. λ. to *ἐδόκει δὲ ὅραμα βλέπειν.*

Give some account of the institution of the feast of unleavened bread. Is the James here mentioned the author of the General Epistle? Give a brief history of the rise and fortunes of Herod Agrippa, and draw a map of the countries included in his dominions at this time.

3. Translate: *Ch.* XVIII., *v.* 12—83.

Γαλλίωνος δὲ ἀνθυπατεύοντος τῆς Ἀχαΐας, κ. τ. λ. to *ἐπιστηρίζων πάντας τοὺς μαθητάς.*

Describe the functions of the *ἀρχισυνάγωγος*. Is Sosthenes mentioned elsewhere in the New Testament? What were the limits of the Roman province of Achaia? Give the grammatical analysis of the following words: *ἠνεσχόμην, ἀπήλασεν, ἀνήχθη, ἀσπασάμενος.*

4. Translate: *Ch.* XXVIII., *v.* 7—16.

Ἐν δὲ τοῖς περὶ τὸν τόπον ἐκεῖνον ὑπῆρχε χωρία, κ. τ. λ. to *σὺν τῷ φυλάσσοντι αὐτὸν στρατιώτῃ.*

Who held the office of *στρατοπεδάρχης* at this time? In what year did St. Paul arrive at Rome, and what is known of his subsequent history? What gave rise to the opinion that he visited Spain? Describe the positon of the island mentioned in the text, and the relative geographical situations of Syracuse, Rhegium and Puteoli.

11. Two known stars are seen at a given place A on the same vertical, when at another place B they are rising together. Find the latitude and longitude of B.

12. A slender ring DEF is attached to another ring ABC by means of a string AD whose length = the radius of ABC; supposing DEF to fall entirely within ABC, determine the tension of the string, when DEF is at rest, force of attraction of $ABC \propto \frac{1}{(\text{dist.})^3}$.

13. If pyramids be formed between three conjugate diametral planes of an ellipsoid and a tangent plane, so that the products of the intercepted portions of the three conjugate diameters, may be the least possible; the volumes of all such pyramids will be equal.

14. Find the nature of the curve in which the tangents, at the extremities of any chord passing through the origin, meet the axis of x in two points equidistant from the origin; and shew from the result that a conic section is a particular case.

15. A ball whose elasticity is e is projected with a velocity V in a direction making an $\angle(\theta+\alpha)$ with the horizon, and rebounds from a plane whose inclination is α which passes through the point of projection. Shew that if R_x, R_{x+1}, R_{x+2} be three consecutive ranges upon the inclined plane after x, $x+1$, and $x+2$ rebounds respectively, $R_{x+2} - (2+e^2) R_{x+1} + e^4 R_x = 0$; and that the sum of all the ranges on the inclined plane before the ball begins to slide down the plane

$$= \frac{2V^2 \sin\beta \sin\theta \cdot \cos(\theta+\beta)}{g \sin\alpha \cdot \cos^2\beta}; \text{ where } \cot\beta = (1-e)\cot\alpha.$$

16. A luminous point is placed in the major axis of an ellipse at a distance u from the centre, shew that the distance u_x of the image from the centre after the xth reflection will be found from the equation

$$\frac{ae-(-1)^x u_x}{ae+(-1)^x u_x} = \left(\frac{1+e}{1-e}\right)^{2x} \frac{ae-u}{ae+u}.$$

17. Two plates of uniaxal crystals are cut perpendicularly to the optic axis, one having its extraordinary ray related to an oblate, the other to a prolate spheroid; compare the magnitudes of the coloured rings, when both plates are placed together between the polarising and analyzing plates, with their magnitudes when each of the plates is placed in the same position separately; the retardation in a plate being expressed by

$$T \frac{c^2-a^2}{2av} \sin^2 i.$$

JUVENAL.

SATURDAY, *Jan.* 15, 1842.

Nine o'clock till Twelve.

Mr. DRAKE, *Examiner.*

FIRST DIVISION.—(A.)

TRANSLATE: *Sat.* X. 56—89.

Quosdam præcipitat subjecta potentia magnæ

to

Tunc de Sejano, secreta hæc murmura vulgi.

Where was Sejanus born? Give a brief account of his career, with the date of his downfall. Where was Capreæ, and for what was it famous? Explain the words *Ex quo suffragia nulli Vendimus.*

Translate: *Sat.* XIII. 143—161.

Rem pateris modicam et mediocri bile ferendam,

to

Dicere te miserum, postquam illinc veneris, aude.

To what crime is allusion made in the words *deducendum corio bovis* &c. &c.? *Custos Urbis.* What is this officer's proper designation?

Translate: *Sat.* XIV. 189—219.

Hæc illi veteris præcepta minoribus: at nunc

to

Exigua et Cereris tangens aramque pedemque

Explain *rubras leges, vitem, locupletem aquilam. Lucri bonus est odor,* &c. Whose saying was this, and on what occasion?

SATURDAY, *Jan.* 15, 1842.

Nine o'clock to Twelve.

Mr. DRAKE, *Examiner.*

(B)

TRANSLATE: *Sat.* X. 257—288.

Incolumi Troja Priamus venisset ad umbras

to

Integer, et jacuit Catilina cadavere toto.

Explain fully the historical allusions, giving the date of each event. Where was Minturna?

Translate: *Sat.* XIII. 120—142.

Accipe, quæ contra valeat solatia ferre,

to

Nos viles pulli nati infelicibus ovis?

Translate: *Sat.* XIV. 304—331.

Misera est magni custodia census.

to

Paruit imperils, uxorem occidere jussus.

To what story is allusion made in the words *Sensit Alexander,* &c. &c. *Effice summam, bis septem,* &c.? What was this law, and when was it passed? *Quadringenta.* What words are to be supplied here? Explain the Roman method of computing money. *Uxorem,* &c. State the circumstances referred to in these words.

ΔΗΜΟΣΘΕΝΟΥΣ ΠΡΟΣ ΛΕΠΤΙΝΗΝ.

WEDNESDAY, *Jan.* 12, 1842.

Half-past Twelve te half-past Three.

W. J. CONYBEARE, *Examiner.*

(A.)

TRANSLATE:

ΕΙ τοίνυν τις ἐάσας ταῦτα αὐτὸ καθ' αὑτὸ ἐξετάσειε, κ. τ. λ. to *τοσούτῳ λῦσαι τὸν νόμον κάλλιον ἢ θέσθαι.*

1. *Εὐηθεία.* Explain the derivation of the word, and illustrate it by words of similar meaning and origin in other languages.

Translate:

Ἀλλὰ νὴ Δί' οὗτοι μόνοι τοῦτο πείσονται, κ. τ. λ. to *μηδὲν ἔχοντες ἐγκαλέσαι; ἀλλ' αἰσχρὸν ἂν εἴη.*

(2) Give some account of Thrasybulus.

Translate:

ΕΠΙΣΧΕΣ. Τοῦτο μέν ἐστιν ἐν τοῖς οὖσι νόμοις κυρίοις ὑπάρχον καλὸν ὦ ἄνδρες Ἀθηναῖοι καὶ σαφές, κ. τ. λ. to *Λαβὲ δ' αὐτὸν τὸν νόμον.*

(3) What were the regulations with regard to proposing laws at Athens?

Translate:

Ὁ τοίνυν νόμος οὗτος ὁ Λεπτίνου οὐ μόνον ὦ ἄνδρες Ἀθηναῖοι τοῦτ' ἀδικεῖ, κ. τ. λ. to *εἰ χαλεπώτερον εἶναι παρ' ὑμῖν δόξει χάριν εὖ ποιήσαντα ἀπαιτεῖν ἢ τὰ δεινότατα ἐργαζόμενον ληφθῆναι;*

(4) Give an account of the object and arguments of the oration against Leptines.

WEDNESDAY, *Jan.* 12, 1842.

Half-past Twelve to half-past Three.

W. J. CONYBEARE, *Ex.*

(B.)

TRANSLATE:

ΤΑΧΑ τοίνυν ἴσως ἐκεῖνο λέγειν ἂν ἐπιχειρήσειε Λεπτίνης, κ. τ. λ. to *ἀλλ' ἐὰν δείξῃ πέντε, ἐγὼ ληρεῖν ὁμολογῶ.*

(1) What peculiar liturgies were the resident aliens at Athens obliged to perform?

Translate:

ΤΑΣ μὲν εὐεργεσίας, ἀνθ' ὧν εὕρετο τὴν ἀτέλειαν ὁ Ἐπικέρδης, κ. τ. λ. to *μάλιστα δ' οἱ παρὰ τὰς χρείας, ὧν εἷς οὗτος ἀνὴρ ὧν φαίνεται.*

(2) What was the proportionate value of a talent, mina, drachma and obolus?

Translate:

ΗΚΟΥΣΑΤΕ μὲν τῶν ψηφισμάτων, ὦ ἄνδρες δικασταί, κ. τ. λ. to *ἃς ἀντὶ τούτων ἔλαβον παρ' ὑμῶν, καὶ δὴ λελυμένας.*

WEDNESDAY, *Jan.* 12, 1842.

Nine o'clock to Twelve.

Mr. ATKINSON, *Examiner.*

FIRST DIVISION.—(B).

TRANSLATE, *Ch.* VIII., *v.* 34—40.

Ἀποκριθεὶς δὲ ὁ εὐνοῦχος τῷ Φιλίππῳ εἶπε, κ. τ. λ. to *ἕως τοῦ ἐλθεῖν αὐτὸν εἰς Καισάρειαν.*

What office did Philip hold in the church? Write down the antient names of Azotus and Cæsarea. What data exist to determine the authorship of the book of the Acts, and the time and place of its composition?

2. Translate: *Ch.* IX., *v.* 20—31.

Καὶ εὐθέως ἐν ταῖς συναγωγαῖς ἐκήρυσσε τὸν Χριστὸν, κ. τ. λ. to *καὶ τῇ παρακλήσει τοῦ ἁγίου πνεύματος ἐπληθύνοντο.*

What is the primary signification of *Χριστός*? Who were the Hellenists? *εἶχον εἰρήνην* ..... With what event does this synchronise? What space of time is included in the term *ἡμέραι ἱκαναὶ*, and how did St. Paul employ it? Enumerate the most remarkable events omitted in the Book of the Acts.

3. Translate: *Ch.* XIII., *v.* 1—12.

Ἦσαν δέ τινες ἐν Ἀντιοχείᾳ κατὰ τὴν οὖσαν ἐκκλησίαν προφῆται καὶ διδάσκαλοι, κ. τ. λ. to *ἐκπλησσόμενος ἐπὶ τῇ διδαχῇ τοῦ Κυρίου.*

What is the import of the term *προφῆται* in the New Testament? Give the derivations of *λειτουργούντων, ῥαδιουργίας, χειραγωγούς, ἀνθύπατος.* Was this last title properly applied to Sergius Paulus? Into what two classes were the Roman provinces divided?

4. Translate: *Ch.* XXV., *v.* 13—22.

Ἡμερῶν δὲ διαγινομένων τινῶν, κ. τ. λ. to *Λύριον, φησὶν, ἀκούσῃ αὐτοῦ.*

State what is known of the Agrippa here mentioned, and trace his descent from Herod the Great. Is the character given of Felix in the Acts corroborated by profane testimony? What kind of franchise did St. Paul possess, and how was it acquired? Give a succinct account of his journey from Cæsarea to Rome, and illustrate your description by a map.

THURSDAY, *Jan.* 13, 1842,

Nine o'clock to Twelve.

FIRST DIVISION.—(A)

I. TRANSLATE:

Το μὲν τοίνυν τῆς πόλεως ἦθος, ὦ ἄνδρες Ἀθηναῖοι, κ. τ. λ. to *βελτίων ἐστὶ τῆς πόλεως τὸ ἦθος.*

(1) What is the meaning of *νόμον τιθέναι* and *νόμον τίθεσθαι*?

II. Translate:

Ἴστε γὰρ δήπου τοῦθ', ὅτι πλείστῳ, κ. τ. λ. to *παρὰ δὲ τὰς δέκα ὡσπερανεὶ τρισχιλίους.*

(2) Whence was Athens principally supplied with corn? Draw a map of that part of Europe and Asia with which it was connected politically or commercially.

III. Translate:

Καὶ μὴν εἴγε ὁμολογῶν ἐκείνους εἶναι, κ. τ. λ. to *τὰ δὲ νῦν ἐπὶ τοῖς νῦν.*

(3) Mention instances of rewards conferred on public benefactors by the early Athenians.

IV. Translate:

Πολλὰ δ' ἄν τις ἔχοι λέγειν ἔτι καὶ διεξιέναι, κ. τ. λ. to *καλῶν καὶ προσηκόντων ὑμῖν ἀγαθῶν ἐλέσθαι.*

(4) Explain the constitution and privileges of the *βουλή* at Athens.

THURSDAY, *January* 13, 1842.

Nine o'clock to Twelve.

(B)

I. TRANSLATE:

Οὕτω τοίνυν τινές, ὦ ἄνδρες Ἀθηναῖοι, κ. τ. λ. to *καὶ περιουσίαις ταῦτα τἀναλώματ' οὐδέν.*

(1) From what sources was the public revenue of Athens derived?

II. Translate:

Εἰ τοίνυν τις ὑμῶν ἐκεῖνο πέπεισται, κ. τ. λ. to *τῶν ἐθελησόντων ἀγαθόν τι ποιεῖν αὐτὴν καταστήσει.*

(2) What is the difference between *ψηφίζειν* and *φέρειν τὴν ψῆφον*?

III. Translate:

Ὃν τοίνυν κακουργότατον οἴονται λόγον εὑρηκέναι, κ. τ. λ. to *ἐὰν ἄρα ταῦτα λεγωσι, τοῦτ' εἶναι δεινὸν νομίζω.*

(3) Which were the sacred liturgies? What is the origin of the word liturgy?

Translate:

Καὶ μὴν οὐδ' ἐκείνου γε ἀποστατέον τοῦ λόγου, κ, τ. λ. to *ἐν οἷς κοσμεῖτε τὰ τῶν ἀγαθῶν ἀνδρῶν ἔργα.*

(4) Give a short history of Athens from the end of the Peloponnesian war to the time of Demosthenes.

ACTS OF THE APOSTLES.

THURSDAY, *January* 13, 1841,

Half-past Twelve to half-past Three.

Mr. M. A. ATKINSON, *Examiner.*

SECOND DIVISION.(A)

I. TRANSLATE:

Στέφανος δὲ πλήρης πίστεως καὶ δυνάμεως ἐποίει τέρατα, κ. τ. λ. to *εἶδον τὸ πρόσωπον αὐτοῦ ὡσεὶ πρόπωπον ἀγγέλου.*

What is mentioned in Scripture respecting Stephen? Explain the meaning of the expression *ἐκ τῆς συναγωγῆς τῆς λεγομένης Λιβερτινων.* Of what number and description of persons did the *συνέδριον* consist? Give the full import of the words *συζητοῦντες, ὑπέβαλον, ἐπιστάντες.* Why did Jesus receive the appellation *ὁ Ναζωραῖος*?

II. Translate:

Ἀναχθέντες δὲ ἀπὸ τῆς Πάφου οἱ περὶ τὸν Παῦλον, κ. τ. λ. to *καθελόντες ἀπὸ τοῦ ξύλου ἔθηκαν εἰς μνημεῖον.*

What were the geographical situations of Paphos, Perga, and the several Antiochs mentioned in the Acts of the Apostles? How long did the people of Israel sojourn in Egypt? and what were the circumstances connected with their being brought out from thence? How was the land of Canaan divided among the Israelites, and what were its political divisions at the time of the Apostles? Who was Pilate, and how came he to have power of life and death at Jerusalem?

III. Translate:

Εἰσελθὼν δὲ εἰς τὴν συναγωγὴν ἐπαῤῥησιάζετο, κ. τ. λ. to *καὶ εὗρον ἀργυρίου μυριάδας πέντε.*

Give the derivations of *ἐπαῤῥησίαζετο, σιμικίνθια, ἐξορκιστῶν, κατακυριεύσας*; and explain the passage *συνεψήφισαν τὰς τιμὰς αὐτῶν,* &c.

IV. Translate:

Γράψας ἐπιστολὴν περιέχουσαν τὸν τύπον τοῦτον· κ. τ. λ. to *λέγειν τὰ πρὸς αὐτὸν ἐπὶ σοῦ. ἔῤῥωσο.*

What was the nature of the government of Judæa at that time? How long did Felix continue governor after the event mentioned above? Give the full meaning of the words *ἐπιστολὴν περιέχουσαν τὸν τύπον τοῦτον*; and shew what is implied by the expression *μαθὼν ὅτι Ῥωμαῖός ἐστι.*

MODERATORS:

The Rev. THOMAS GASKIN, M.A., *Fellow of Jesus College.*

DUNCAN FARQUHARSON GREGORY, M.A., *Fellow of Trinity College.*

EXAMINERS:

The Rev. ALEXANDER THURTELL, M.A., *Fellow of Gonville and Caius College.*

RICHARD POTTER, M.A., *Fellow of Queens' College.*

Problems.

WEDNESDAY, *January* 5, 1842,

Nine o'clock to half-past Eleven.

Mr. GREGORY, *Moderator.*—Mr. THURTELL, *Examiner.*

N. B. The Differential Calculus not to be used.

1. IF a straight line fall on two parallel straight lines it makes the alternate angles equal.

2. If a straight line touch a circle, and if from the point of contact a straight line be drawn cutting the circle, the angles which it makes with the touching line are equal to the angles in the alternate segments of the circle.

3. Prove the rule for pointing in the division of decimals.

Ex. Divide 95.88 by 4700 and also by .068.

4. Divide $x^4 - ax^3 + 2a^3x - 4a^4$ by $x^2 - 2a^2$; and find the condition that $ax^2 + bx + c$ may be a complete square.

5. If $a : b :: a_1 : b_1 :: a_2 : b_2$, shew that

$$a : b :: a + a_1 + a_2 : b + b_1 + b_2,$$

and $a : b :: \sqrt{a^2 + a_1^2 + a_2^2} : \sqrt{b^2 + b_1^2 + b_2^2}$.

6. Solve the simultaneous equations

$$ax + by = c, \qquad a_1x + b_1y = c_1.$$

What form do the values of x and y take when

$$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1},$$

and what does it indicate?

7. Shew that the sum of a fraction and its reciprocal is always greater than 2. How can we, consistently with this truth, assume that

$$2\cos\theta = x + \frac{1}{x}?$$

8. Prove that $\cos(A - B) = \cos A \cos B + \sin A \sin B$, and that $(\cos A)^2 - (\sin B)^2 = \cos(A + B)\cos(A - B)$.

9. Express the cosine of half the angle of a plane triangle in terms of the sides. Are the two values given by the formula admissible?

10. Find the equation to a straight line which passes through a given point, and cuts off a given portion from the axis of x.

11. Find the polar equation to the parabola, the focus being the pole; and shew that $SP = SA\,(\text{cosec}\,\theta)^2$, where θ is the angle which the tangent at P makes with the axis.

12. Prove that in the ellipse $SP.HP = CD^2$.

13. Shew how to transform an equation into one of which the roots are the same as those of the original equation but with opposite signs.

Solve the equation $x^4 + 3x^3 + 2x^2 + 9x - 3 = 0$, of which two roots are equal but of opposite signs.

14. Prove that the sum of the angles of a spherical triangle is always greater than two right angles, and less than six right angles.

15. If a, b be the sides, c the hypotenuse of a right-angled spherical triangle, shew that $\cos c = \cos a \cos b$. Thence deduce the corresponding property of right-angled plane triangles.

16. Prove that

$$\log_a(1 + x) = \frac{1}{\log_e a}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.\right)$$

Having given $\log_{10} 2 = .301030$, find $\log_{10} .0025$.

MONDAY, *January* 10, 1842,

One o'clock to Four.

Mr. GASKIN, *Mod.*—Mr. THURTELL, *Ex.*

1. DEFINE discount, and determine the discount on £120. 16s. 8d., due ten months hence, at $4\frac{1}{2}$ per cent.

2. Explain the operation of the common pump, and find the force to be overcome at any part of the stroke.

3. Find the equations to the asymptotes of the curve $(x-2a)y^2=x^3-a^3$, and shew that the curve cuts two of the asymptotes at points whose co-ordinates are

$$-\frac{a}{3}, \quad \pm\frac{2a}{3}.$$

4. Shew that the moment of inertia of a body, round an axis passing through any point, is equal to the moment of inertia round an axis parallel to it, through the centre of gravity, added to the product of the mass of the body and the square of the distance between the two axes.

Ex. Find the moment of inertia of a cylinder round a tangent to one of its circular ends.

5. Point out the effect of parallax on the circumstances of a solar eclipse, and shew how to construct for the places upon the Earth's surface at which a given solar eclipse is visible.

6. Shew that

$$\Delta^n \cos \pi x = 2^n \cos \pi (x+n),$$

$$\Delta^n u_x v_x = u_x \Delta^n v_x + n \Delta u_x \Delta^{n-1} v_{x+1} + \frac{n(n-1)}{1.2} \Delta^2 u_x \Delta^{n-2} v_{x+2} + \&c.,$$

and that $\Delta^n o^n$, $\Delta^n o^{n+1}$, $\Delta^n o^{n+2}$...form a recurring series.

7. Find the horary motion of the Moon's nodes in a circular orbit; and shew that the mean horary motion of the nodes is half the horary motion when the Moon is in syzygy.

8. Explain how the attraction of the Moon on the Earth, each being supposed stationary, would raise the waters of the ocean on the side of the Earth nearest the Moon, and also on the side farthest from it.

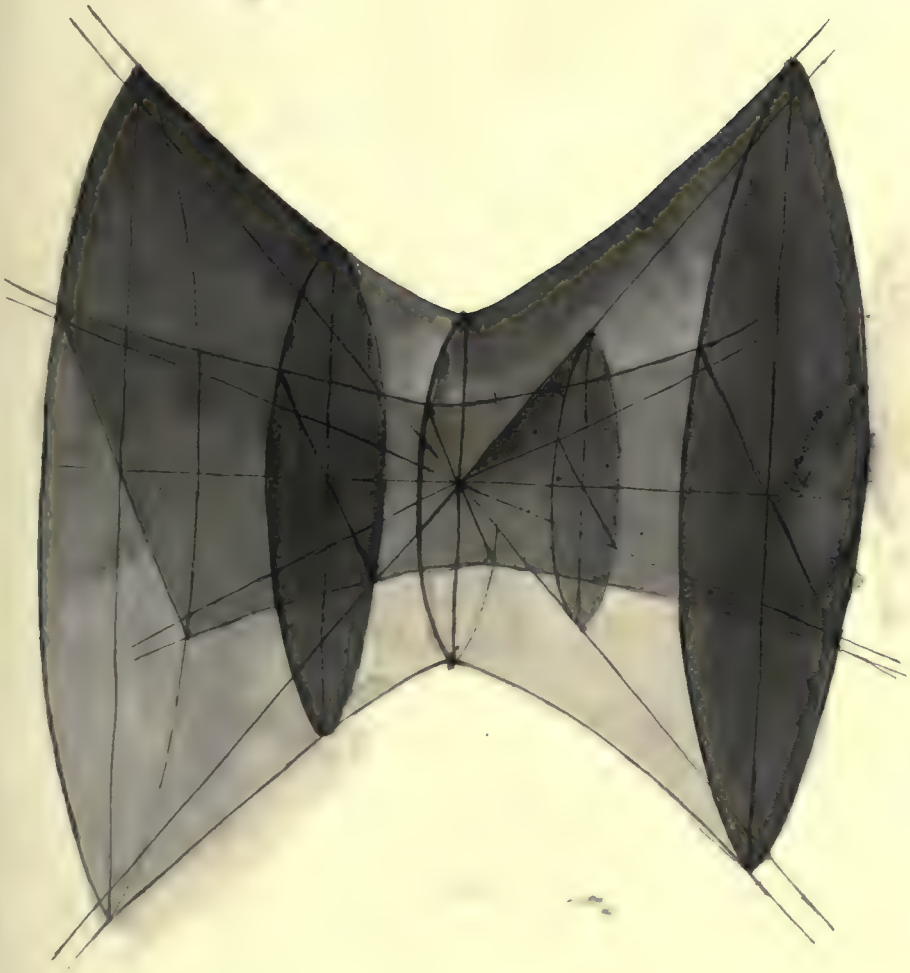
Assuming the expression for the height of the tide at a given time and place, find the time of high tide.

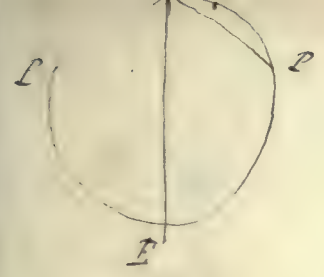
9. An event having happened m times and failed n times in $m+n$ trials, find the probability that it will happen in the next trial, the probability of the simple event being unknown.

10. Find the whole accelerating force which acts on the Moon in a direction perpendicular to the plane of the ecliptic.

11. State and prove the principle of Vis Viva, and shew that a body will pass through a position of stable or unstable equilibrium according as the Vis Viva is a maximum or a minimum.

12. Find the condition that $\int V dx$ may be a maximum or a minimum when $\int U dx$ is given. Ex. Find the curve which generates, by its revolution round the axis of x, the solid of least surface when the area is given.



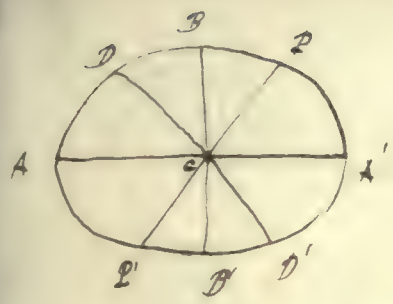


Let L P P' be the two forces & the pt of applic
of the 2 forces passing through P & P' since the two
forces are equal their resultant must bisect the
∠ P C P'. Let the resultant & E produced meet circumf in T.

∴ ∠ E C P = ∠ E C P' ∴ circumf P T = circumf P' P'

But C is any point in the circumf ∴ the resultant of each
two of the forces would bisect the circumf P E P' in E

∴ they would all pass through the same point E



Let AA' BB' be the principal axes of the ⊙

∴ they are at right angles to each other; also

let $CA' = a$, $CB = b$ be the semi axes of the ⊙

let CP, CD be any other semi-conjugate diameters, $CP = a'$, $CD = b'$

then $a'b' \sin\gamma = ab$, if angle $PCD' = \gamma$

also if $x'y'$ be co-ord^{s} of P, $\frac{ay'}{b}$, $\frac{bx'}{a}$ are co-ord^{s} of D.

$$\therefore CP^2 + CD^2 = x'^2 + y'^2 + \frac{a^2 y'^2}{b^2} + \frac{b^2 x'^2}{a^2} = a^2 + b^2$$

$$\therefore a'^2 + b'^2 = a^2 + b^2$$

Now if $\angle PCD$ is a right angle, or the conjugate diameters PP', DD' are rectangular we have $a'b' = ab$

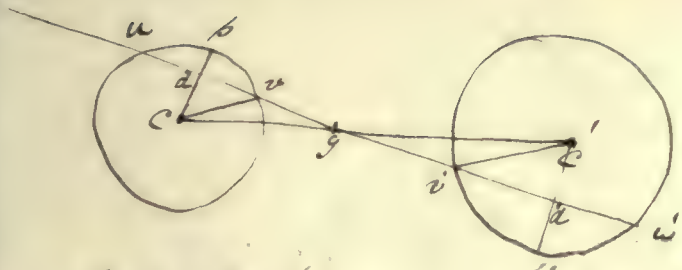
also in all positions $a'^2 + b'^2 = a^2 + b^2$

∴ in this case $a' = a$, $b' = b$

or the conjugate diameters coincide with the principal axes of the ⊙, which are therefore the only conjugate diam^{rs} which cut each other at right angles.

(by CP, CD are any conjugate diameters.)

Senate House. Smth.



Let A be the given point. cc' the centres of the circles. a and a' the radii be r, r' respectively.

Join cc', and divide it in g so that $\frac{cg}{c'g} = \frac{r}{r'}$

Join Ag and produce it to cut the other circumf., this shall be the line required.

From c & c' let fall $cv, c'v'$ perp. to Ag. join $cv, c'v'$.

then the sides of the $\triangle$ cvg are evidently $\parallel$ (whose of the $\triangle$ $c'v'g$

$\therefore$ $\angle$ $cvd = \angle$ $c'v'd'$ (for they are both external angles

also $\angle$'s at d and d' are right angles

$\therefore$ $\angle dcv = \angle d'c'v'$

$\therefore$ $\triangle$ cvd is equiangular to $\triangle$ $c'v'd'$

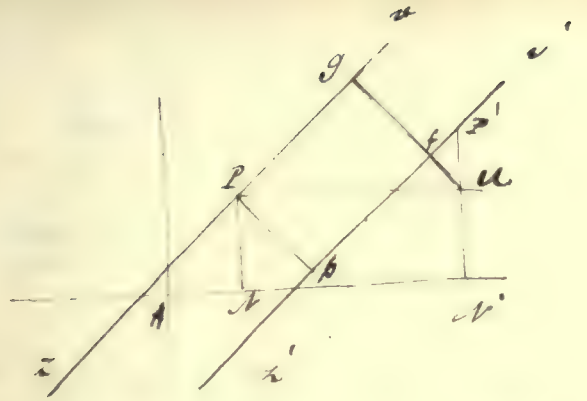
$\therefore$ it has its sides $\parallel$.

$$\frac{r}{r'} \quad \frac{cv}{c'v'} = \frac{vd}{v'd'}.$$

But $vu = 2\,dv$, and $v'u' = 2\,d'v'$

$$\therefore \frac{vu}{v'u'} = \frac{r}{r'}$$

Lenata House



Let Oz, Oz' be the two fixed lines, PP' the two points

And the origin $AN = b$, $NP = a$ co-ords of P.

$AN' = b'$ $NP' = a'$ co-ords of P'

Through P draw $Pu \parallel$ to AN'

and through u P draw indef. $Pb \perp$ to Oz.

Then $Pp = gf = gu - gf = Pu \cos\theta - P'u \sin\theta$

$= (b' - b)\cos\theta - (a' - a)\sin\theta$

Let ab. $a'b'$ $a''b''$ $a'''b'''$ be four points through which the sides of a rectangle pass. and let θ be the angle which the $\parallel$ through PP' makes with Oz.

$\therefore$ the $\perp$ distance between them =

$(b'' - b)\cos\theta - (a'' - a)\sin\theta$.

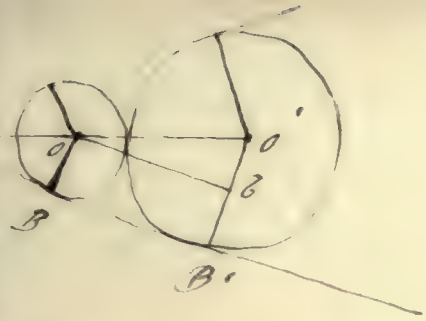
also the $\perp$ distance between the sides passing through $P'P'''$ are

$(b' - b''')\sin\theta - (a' - a''')\cos\theta$.

also $\{(b' - b''')\sin\theta - (a' - a''')\cos\theta\}\{(b'' - b')\cos\theta - (a'' - a')\sin\theta\} = A$

from which θ is easily determined when the rectangle is completely determined.

Lenate hence.



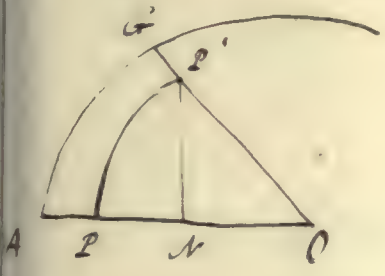
Let A be the point where the tangent meets
O'.O the centres of the Os.

Join OA which will evidently pass through O. for the circles are symmetrically situated with respect to the tangent.

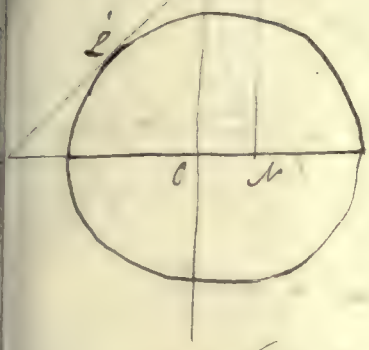
Draw $OB \parallel$ to AB' $\therefore \angle O'OB = \angle OAB' = \frac{\theta}{2}$.

$\sin O'OB = \frac{O'B}{OO'} = \frac{a-b}{a+b}$ $\cos O'OB = \sqrt{1-\sin^2 O'OB} = \frac{2\sqrt{ab}}{a+b}$.

$\therefore \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{4(a-b)\sqrt{ab}}{(a+b)^2}$



Sin A, 4 mule



Let the coordts of Q be $ON = a$, $NQ = b$.
and let QT be the line touching the circle in $Q'(x,y)$
then QT is evidently a tangent at Q' on the ⊙ of ⊙.

(1) $by + ax = r^2$ for it passes through the pt Q.

but x, y being pts in the ⊙ must answer the conditions $x^2 + y^2 = r^2$ (2)

Now putting $a = 0$ in eqn (1) we have

$$by = r^2 \quad \therefore \quad b' = \frac{r^2}{y} = c$$

$$\therefore r^2 = yc \text{ or } y = \frac{r^2}{c}$$

$$\text{also } x^2 = r^2 - y^2 = r^2 - \frac{r^4}{c^2} = r^2\left\{1 - \frac{r^2}{c^2}\right\}$$

$$\therefore x = r\sqrt{1 - \frac{r^2}{c^2}}$$

∴ substituting these values for x & y in eqn (1) we have

$$\frac{br^2}{c} + \frac{ar\sqrt{c^2 - r^2}}{c} = r^2$$

$$br^2 + a\sqrt{c^2 - r^2} = cr \quad \therefore c = b + \frac{a}{r}\sqrt{c^2 - r^2}$$

$$(b - c)r = -a\sqrt{c^2 - r^2}$$

$$b^2 - 2bc + c^2 r^2 = a^2 c^2 - a^2 r^2$$

$$(b^2 + a^2)r^2 = (a^2 - r^2)c^2 + 2br^2 c$$

$$c^2 + \frac{2br^2}{a^2 - r^2}c - \frac{(b^2 + a^2)r^2}{a^2 - r^2} = 0 \text{ the req}^d \text{ eq}^n.$$

Senate House.

Let $xy' + xx' = r^2$
$xy'' + xx'' = r^2$ } be the eqns to any two tangents

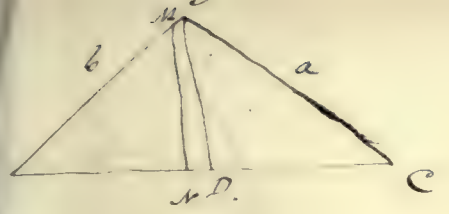
the positions of the satisfy each of by these eqns at

$\frac{r^2}{y'}$ and $\frac{r^2}{y''}$ respectively.

$\therefore \frac{r^2}{y'} \times \frac{r^2}{y''} = \frac{r^4}{y'y''} = P$, a constant

$\therefore r^4 = P\, y'y''$ is the req'd eq'n.

$(x'y')$ $(x''y'')$ being the co-ords of the points of contact of the tangents.



Let B be the right angle.

BD ⊥ to AC. MN // to BD bisects the Δ.

Let AC = x. NM = y ∴ x : y :: AD : DB :: b : a.

Or on ax = by $x = \frac{b}{a} y$.

also area of Δ ABC = $\frac{ab}{2}$

∴ area of Δ AMb by the supposit. = $\frac{ab}{4}$

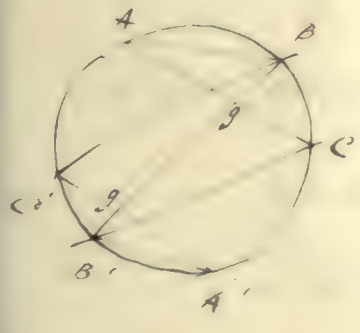
$\therefore xy = \frac{ab}{4}$ $x = \frac{ab}{4y} = \frac{b}{a} y$

$a^2 = 4y^2$

$a = 2y \therefore y = \frac{a}{2}$

also $x = \frac{b}{2}$ and $xy = \frac{ab}{4}$.

Make Ab = $\frac{AB}{2}$. and erect the ⊥ NM which will bisect the Δ.



AA' // to AB'' AC // to AC''

∴ ∠ BAC = ∠ B'AC'

∴ BB'C' = ∠ BB''C'' } for they are upon the same circumf.

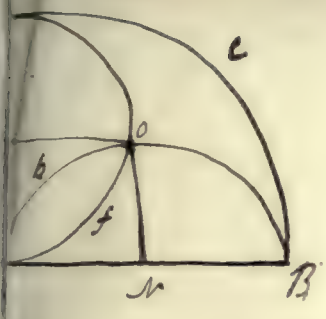
also ∠ Bg'C = ∠ Bg C' for AC. A'C' are //

∴ Δ C'g B is equiangular with Δ Bg'C

∠ ACB' = ∠ A'C'B'

and B'C' is // to BC'

Lenatte Lange



Take A for the origin and a Line

$Oh^2 = 2a\,Ah + Ah^2$

also $Oh'^2 = 2a\,Ah' + Ah'^2$

But Ah' is a $\square$ $\therefore Ah' = hO$ and $Ah = h'O$

$\therefore OM^2 = 2a\,OM' + OM'^2$

But OM' is evidently $= OM$ $\therefore$

$2OM^2 = 2a\,OM'$ $OM = a$. $\therefore x = y = a$. if $AB = 2a$

Now area of $\square$ $BG = 4a^2\frac{\pi}{4} = a^2\pi$. (1)

area of $\square$ $h'OA = \frac{a^2\pi}{4}$

area of $\square$ $hONA = a^2$ $\therefore$ $a^2 - \frac{a^2\pi}{4}$ = area of ofA

$a^2 - 2\left(a^2 - \frac{a^2\pi}{4}\right) = \frac{a^2\pi}{2} - a^2$ = area of $obfA$. (2)

also area of $\square$ $GOA = \frac{a^2\pi}{2}$

area of $OfAB = a^2$

$\therefore$ area of $GOBA = a^2 + \frac{a^2\pi}{2}$

But area of the quad^{l} $GB = a^2\pi$

$\therefore$ area of $GOBc = a^2\pi - \left\{a^2 + \frac{a^2\pi}{2}\right)$

$= \frac{a^2\pi}{2} - a^2$

= area of $obfA$.

Lenete Hanna.

$u \cdot \nabla h = X \quad \therefore u \cdot \nabla \log_e \rho = X$

$u \cdot \nabla k = Y \quad u \cdot \nabla \log_e p = Y$

$u \cdot \nabla p = Z \quad u \cdot \nabla \log_e p = Z$

$\nabla_x \rho = e^X \quad \nabla_y \rho = e^Y \quad \nabla_z \rho = e^Z$

$\nabla_x \nabla_y \rho = \nabla_y \nabla_x \rho \quad \therefore \nabla_x Y = \nabla_y X$

$\nabla_z \nabla_x \rho = \nabla_x \nabla_z \rho \quad \nabla_z X = \nabla_x Z$

$\nabla_y \nabla_z \rho = \nabla_z \nabla_y \rho \quad \nabla_y Z = \nabla_z Y$

$X(\nabla_z Y - \nabla_y Z) + Y(\nabla_z X - \nabla_x Z) + Z(\nabla_y X - \nabla_x Y) = 0$

If the pressure $= c$ at any point we have $\rho = c$ & the density $= c$ & the temperature $= c$ & the pressure is a function of x only.

The latter eqn is

$\nabla_x \rho + \nabla_z \rho$

$y = \xi^{\frac{1}{1-\log_\xi x}}$ $\therefore \log_\xi y = \frac{1}{1-\log_\xi x}$, $\log_\xi x = 1 - \frac{1}{\log_\xi y} =$

$$= \frac{\log_\xi y - 1}{\log_\xi y} = \frac{1}{(1-\log_\xi y)\log_b y} \quad (1)$$

also $x = \xi^{\frac{1}{1-\log_\xi y}}$ (B)

$\therefore \frac{1}{1-\log_\xi y} = \log_\xi x$ / by substitution / $\log_\xi x = \frac{\log_\xi x}{\log_\xi y}$ (2)

also from (B) $1 - \log_\xi y = \frac{1}{\log_\xi x}$ $\therefore \log_\xi y = \frac{1-\log_\xi x}{\log_\xi x}$ (3)

from (11) we have $\log_\xi y = \frac{\log_\xi x}{\log_\xi x} = \frac{1-\log_\xi x}{\log_\xi x}$ from (3)

$$= -\frac{\log_\xi x}{1-\log_\xi x}$$

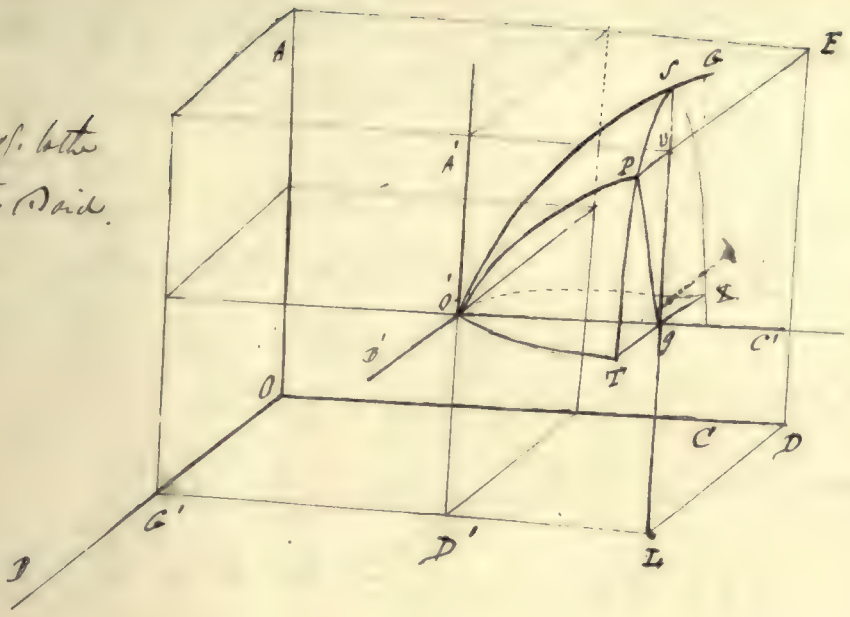
$$\therefore \frac{1}{\log_\xi x} = -\frac{1}{1-\log_\xi x}$$

$$\log_\xi x = \frac{1}{1-\log_\xi x}$$

or $x = \xi^{\frac{1}{1-\log_\xi x}}$.

Lenita Hawks.

find the eq^{n}. to the surface of the Solid.



P.S. If the revolution take place about the axis of z OA' the eq^{n}. to the surface will be

$$(z-z')^2 = 4a\sqrt{(x-x')^2+(y-y')^2}$$

Let O be the origin OA, OB, OC the axes of z, y, and x.

$OD = x$, $DE = y$. $EP = z$ co-ords of P.

O' the new vertex of the parabolic curve OG,

$O'A'$, $O'C'$ the axes of the $\bigcirc$ respectively $\parallel$ to OA, OC.

Let $O'D' = x'$, $D'G' = y'$ $OG' = z'$ be co-ords of O'

Let $O''P$ be the position of the $\bigcirc$ in any other position. TS a plane $\parallel$ to xy. Sg a section made with this plane by a plane passing through OA', OC', Pv in Pg.

Then from $\triangle gPv$. right angled at v. $Pg^2 = Pv^2 + vg^2$

and $vg = vL - gL = DE - O'D' = x - x'$

also $Pv = PE - vE = PE - OG' = z - z'$

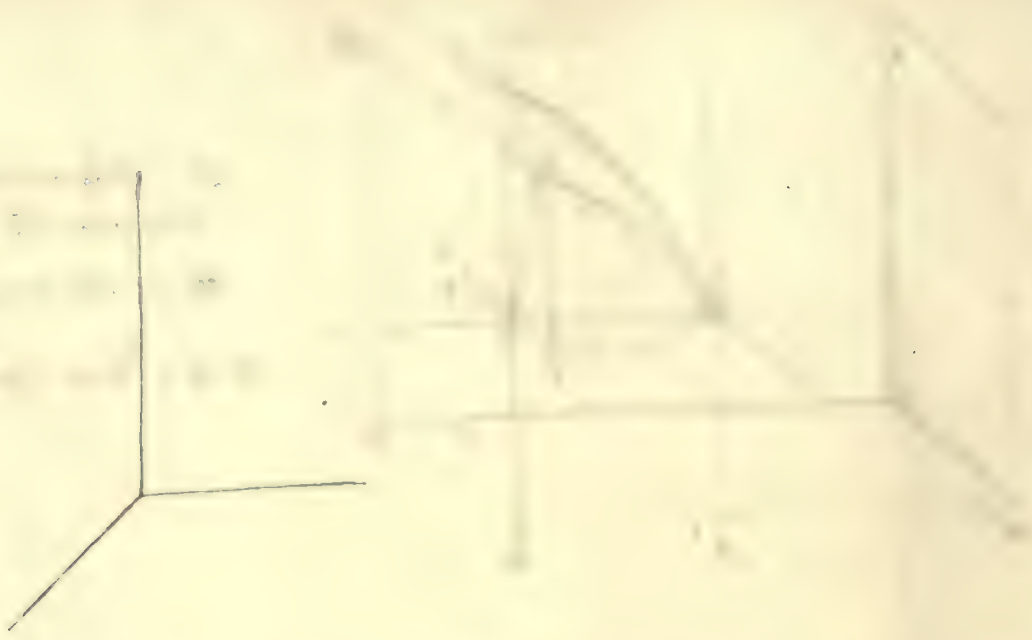
$\therefore Pg^2 = (z-z')^2 + (y-y')^2$

also since P is a point $\in$ the parabola we have

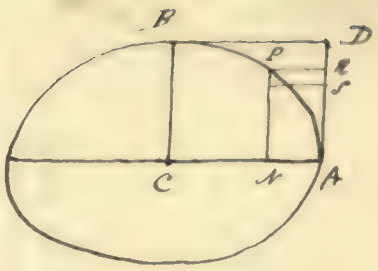
$Pg^2 = 4a\,O'g$; and $O'g = D'L = GL - G'D' = OD - G'D'$

$= x - x'$

$\therefore 4a(x-x') = (z-z')^2 + (y-y')^2$ is the eq^{n}. to any $\bigcirc$ revolving round $O'C'$ and therefore the eq^{n}. of the Sd. described by the revolution of



Let ABC be an elliptic area cutoff by the semi-diam CB CA. Complete the ∥gm CP. ∴ AD is a tangent at A.



Let CA = a. CB = b. AD divided into n = spaces. RS the r'th of these spaces. Let PR = y^r, AR = x = $\frac{rb}{n}$.

Then for eq. to the ellipse

$$AC^2.AR^2 + CB^2.(AC - PR)^2 = AC^2.CB^2$$

$$a^2.\frac{r^2b^2}{n^2} + b^2(a - y^r)^2 = a^2b^2 \;;\; \frac{a^2r^2}{n^2} + (a - y^r)^2 = a^2$$

$$(a - y^r)^2 = a^2\left\{1 - \frac{r^2}{n^2}\right\} \quad \text{or} \quad y^r = a\left\{1 - \sqrt{1 - \frac{r^2}{n^2}}\right\} = a - a\sqrt{1 - \frac{r^2}{n^2}}$$

Now area of the part BAD = $b\sum_{r=1}^{n} i.\frac{y^r}{n}$.

$$= b\sum_{r=1}^{n}\frac{na}{n} - \frac{ba}{n}\sum\sqrt{1 - \frac{r^2}{n^2}}$$

by giving to r the several values 1. 2. 3 —

Now we have $\left(1 - \frac{1}{n^2}\right)^{\frac{1}{2}} = 1 - \frac{1}{2n^2} + \frac{1\times 1}{2\cdot 4}\cdot\frac{1}{n^4}$ —

$$\left(1 - \frac{4}{n^2}\right)^{\frac{1}{2}} = 1 - \frac{4}{2n^2} - \frac{1}{8}\cdot\frac{16}{n^4}$$

$$\left(1 - \frac{9}{n^2}\right)^{\frac{1}{2}} = 1 - \frac{9}{2n^2} - \frac{1}{8}\cdot\frac{81}{n^4}$$

∴ sum of n terms $\times\frac{a}{n} = \frac{a}{n}\left\{n - \frac{1}{2n^2}\left\{1 + 2^2 + 3^2 - \text{to } n \text{ terms}\right\} - \frac{1}{8n^4}\left\{1 + 2^4 + 3^4 \text{ to } n \text{ terms}\right\}\right.$

$$= \frac{a}{n}\left\{n - \frac{1}{2n^2}\left\{\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right\} - \frac{1}{8n^4}\left\{\frac{n^5}{5} + \text{&c.}\right\}\right\}$$

$$\frac{a}{n}\left\{\frac{5n}{6} - \frac{n}{40} - \text{terms not involving } n = P.\right\}$$

$$\frac{a}{n}\left(\frac{194n}{240} - P.\right) = \frac{194a}{240} - \frac{P}{n} \quad \therefore \sum\frac{a}{n}\sqrt{1 - \frac{r^2}{n^2}} = \frac{194n}{240}$$

∴ area of BAD = $b\left\{a - \frac{194a}{240}\right\} = \frac{ab.46}{240} = \frac{23}{120}.ab.$

∴ Elliptic area = $\frac{97}{120}.ab. = \frac{4}{5}ab$ nearly.

(16) If $\tan\theta = n \tan\varphi$, then we shall have

$$\theta = \varphi + \left(\frac{n-1}{n+1}\right)\sin 2\varphi + \frac{1}{2}\left(\frac{n-1}{n+1}\right)^2 \sin 4\varphi + \text{etc.}$$

$$\left(\frac{\varepsilon^{2\theta\sqrt{-1}}-1}{\varepsilon^{2\theta\sqrt{-1}}+1}\right) = n\left(\frac{\varepsilon^{2\varphi\sqrt{-1}}-1}{\varepsilon^{2\varphi\sqrt{-1}}+1}\right)$$

$$-\varepsilon^{2\theta\sqrt{-1}} = \frac{(n+1)\varepsilon^{\varphi\sqrt{-1}} - (n-1)\varepsilon^{-\varphi\sqrt{-1}}}{(n-1)\varepsilon^{\varphi\sqrt{-1}} - (n+1)\varepsilon^{-\varphi\sqrt{-1}}}$$

$$\varepsilon^{2\theta\sqrt{-1}} = \frac{(n-1)\varepsilon^{-\varphi\sqrt{-1}} - (n+1)\varepsilon^{\varphi\sqrt{-1}}}{(n-1)\varepsilon^{\varphi\sqrt{-1}} - (n+1)\varepsilon^{-\varphi\sqrt{-1}}}$$

$$\text{or} \quad \varepsilon^{2\theta\sqrt{-1}} = \frac{\varepsilon^{\varphi\sqrt{-1}}\left\{(n-1)^{-2\varphi\sqrt{-1}} - (n+1)\right\}}{(n-1)\varepsilon^{2\varphi\sqrt{-1}} - (n+1)}$$

$$\varepsilon^{2\theta\sqrt{-1}} = \varepsilon^{2\varphi\sqrt{-1}}\left\{\frac{\frac{n-1}{n+1}\varepsilon^{-2\varphi\sqrt{-1}} - 1}{\frac{n-1}{n+1}\varepsilon^{2\varphi\sqrt{-1}} - 1}\right\}$$

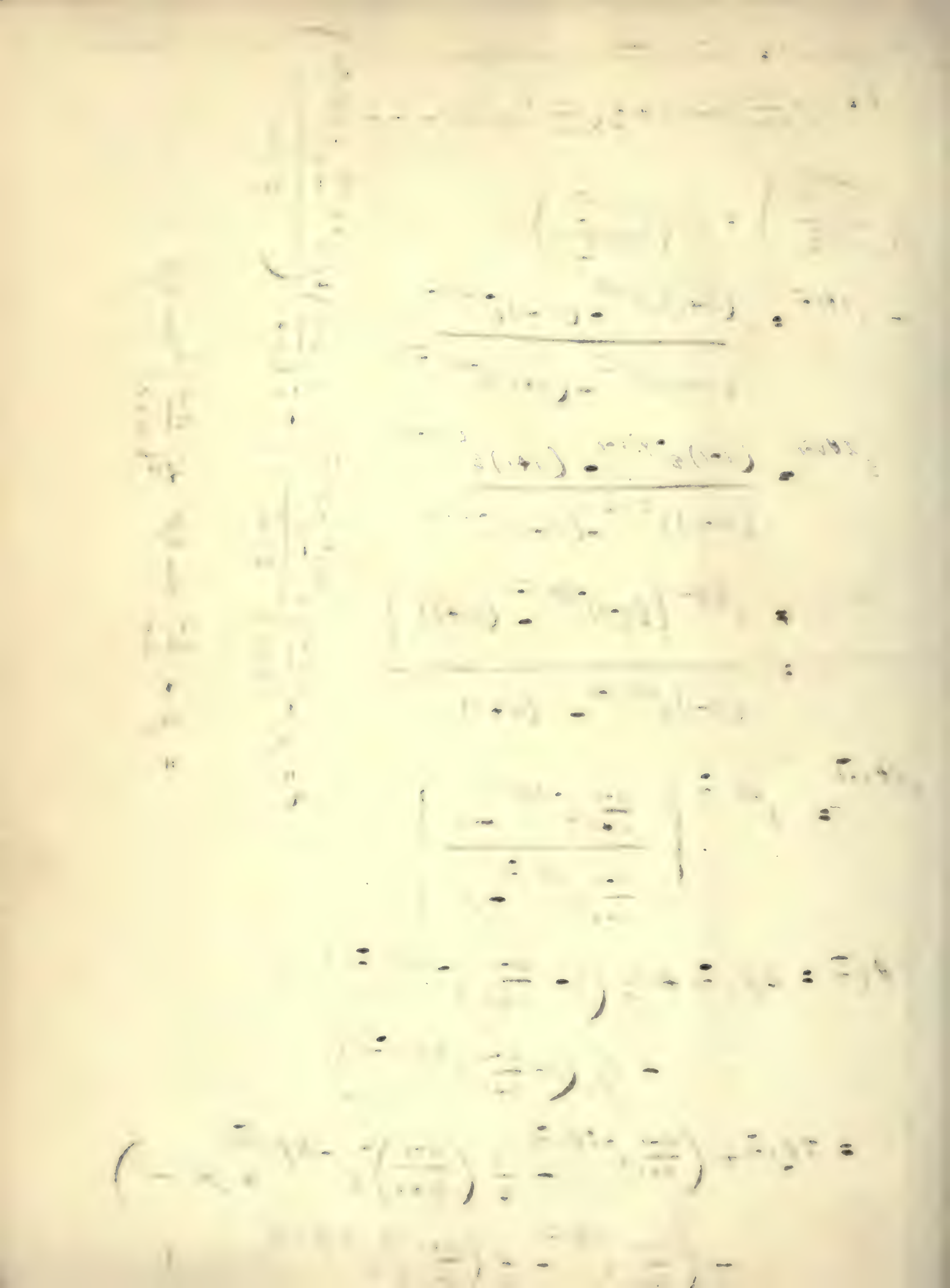
$$2\theta\sqrt{-1} = 2\varphi\sqrt{-1} + \log\left(1 - \frac{n-1}{n+1}\varepsilon^{-2\varphi\sqrt{-1}}\right) - \log_\varepsilon\left(1 - \frac{n-1}{n+1}\varepsilon^{2\varphi\sqrt{-1}}\right)$$

$$= 2\varphi\sqrt{-1} + \left(\frac{n-1}{n+1}\varepsilon^{-2\varphi\sqrt{-1}} - \frac{1}{2}\left(\frac{n-1}{n+1}\right)^2 \varepsilon^{-4\varphi\sqrt{-1}} + \text{etc.}\right)$$

$$- \left(\frac{n-1}{n+1}\varepsilon^{2\varphi\sqrt{-1}} - \frac{1}{2}\left(\frac{n-1}{n+1}\right)^2 \varepsilon^{4\varphi\sqrt{-1}} - \text{etc.}\right)$$

$$\therefore \theta = \varphi + \frac{n-1}{n+1}\left\{\frac{\varepsilon^{2\varphi\sqrt{-1}} - \varepsilon^{-2\varphi\sqrt{-1}}}{2\sqrt{-1}}\right\} + \frac{1}{2}\left(\frac{n-1}{n+1}\right)^2\left(\frac{\varepsilon^{4\varphi\sqrt{-1}} - \varepsilon^{-4\varphi\sqrt{-1}}}{2\sqrt{-1}}\right) - \text{etc.}$$

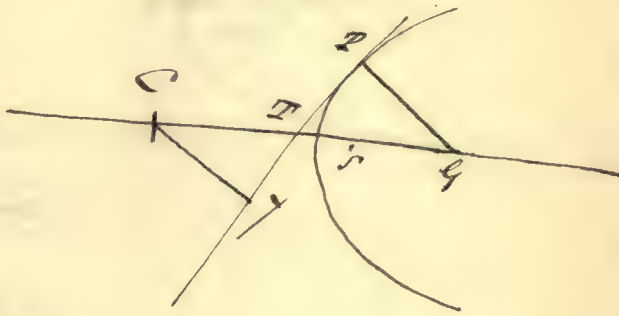
$$= \varphi + \frac{n-1}{n+1}\sin 2\varphi + \frac{1}{2}\left(\frac{n-1}{n+1}\right)^2 \sin 4\varphi - \text{etc.}$$



If P be a point in an $\mathcal{H}$ whose ordinate $= \left(\frac{BC}{SC}\right)^{\frac{1}{2}}$
& Cy be a ⊥ on the tangent from the centre then $Py = SC$

Let $\angle PTx = \theta$

$Py = CG \cos\theta$, PG being a
normal at P.



or since if x, y be co-ords of P,
$CG^2 = e^2x^2$, and $\tan\theta = \frac{b^2x}{a^2y}$

$$Py^2 = \frac{a^4e^4x^2y^2}{a^4y^2 + b^4x^2} \qquad (1)$$

but since $y^2 = \frac{b^3}{ae}$ $\quad x^2 = \frac{a^2}{b^2}(b^2+y^2) = \frac{a^2}{ae}(b+ae)$

$$\therefore a^4y^2 + b^4x^2 = \frac{a^4b^3 + a^2b^4(b+ae)}{ae} = \frac{a^2b^3(a^2+b^2+bae)}{ae}$$

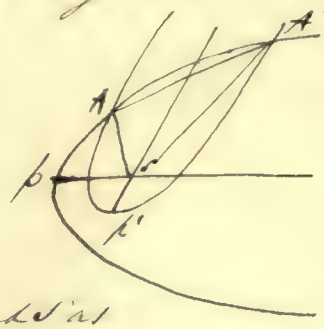
$$= a^2b^3(b+ae)$$

and $a^4e^4x^2y^2 = a^2e^2a^2b^3(b+ae)$

$$\therefore Py^2 = ae^2$$

$$Py = ae = SC.$$

If two ⊙'s have a common pt. A, and a common focus S,
and their axes inclined to AS at ∠s the sum of which = 2 right angles
their other point of intersection will be in a line through A ⊥ to AS



Let ∠ ASP = α, ∠ ASP' = β
P and P' being the vertices of the ⊙'s.
Then α + β = π
and taking AS as the polar radius and S as
the pole, the = ns to the ⊙'s will be

$$r = \frac{2a}{1 + \cos(\alpha + \theta)} \qquad \text{——— (1)}$$

$$r = \frac{2a'}{1 + \cos(\beta + \theta)} \qquad \text{——— (2)}$$

Hence if A' be drawn ⊥ to AS cutting the first ⊙ in A'
we have (∠ ASA' being = θ)

$$\cos\theta = \frac{AS}{A'S} = \frac{1 + \cos(\alpha + \theta)}{1 + \cos\alpha}$$

$$\text{or } \cos\theta = 1 - \sin\alpha \sin\theta \qquad \text{——— (3)}$$

similarly if θ' be the corresponding ∠ for the
second ⊙ $\cos\theta' = 1 - \sin\beta \sin\theta'$

$$\text{or } \cos\theta' = 1 - \sin\alpha \sin\theta' \quad (4) \text{ since } \alpha + \beta = \pi$$

Hence comparing (3) and (4) θ' = θ, or the ⊙'s
intersect in the pt. A.

Find the locus of the points of intersection of tans. drn ⊥ ly which meet at rt ∠s.

If α be the tan. of the ∠ wh one of the tans. makes with the axis,

& $y^2 = 4ax$ the eqn of the ⊥

the eqn of the tan. will be $y = \alpha x + \frac{a}{\alpha}$

the eqn of the tan. at rt ∠s to this is

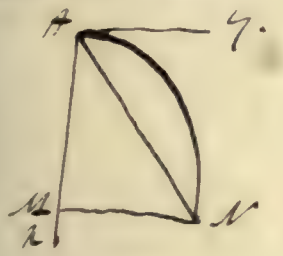
$$y = -\frac{1}{\alpha}x - a\alpha$$

∴ subtracting

$$(\alpha + \tfrac{1}{\alpha})(x + a) = 0$$

or $x + a = 0$ the eqn of the req'd locus.

a Δ has its vertex in the surface of a fluid and its axis vertical, find the cent. of pressure when cut off by a line inclined at 45° to the axis, and passing thro' the vertex.



Let $\left\{\begin{array}{l} y^2 = 4ax. = m \text{ to } \Delta \\ y = x = m \text{ to line} \end{array}\right.$

Then for the pts. of intersection of the Δ and line

$\sqrt{mx} = x$ or $\sqrt{x} = \sqrt{m}$ $\therefore x = m = AM = AN$

$$\therefore \bar{x} = \frac{\int_0^m \int_x^{\sqrt{mx}} x\,dy\,dx}{\int_0^m \int_x^{\sqrt{mx}} dy\,dx} = \frac{\int_0^m x.\sqrt{m}.x^{\frac{1}{2}} - x^2}{\int_0^m x.\sqrt{m}x^{\frac{1}{2}} - x^2} = \frac{\frac{2}{7}m^4 - \frac{m^4}{4}}{\left(\frac{2}{5} - \frac{1}{3}\right)m^3}$$

$$= \frac{15}{28}\,m$$

$$\text{also } \bar{y} = \frac{\int_0^m \int_x^{\sqrt{mx}} y\,dy\,dx}{\frac{1}{15}m^3} = \frac{15}{2m^3}\int_0^m \left(mx^2 - x^3\right)$$

$$= \frac{15m}{2}\left(\frac{1}{3} - \frac{1}{4}\right)\frac{15}{24}m = \frac{5m}{8}$$

$$\therefore \bar{x} = \frac{15}{28}m, \quad \bar{y} = \frac{5}{8}m.$$

where m is derived from $= m$ to the Δ

$$y^2 = mx.$$

If a be the natural and ρ the stretched length of the string, the tension will be $\mu(\rho - a)$ by Hooks principle

$$\therefore T = \mu(\rho - a).$$

Also in a $\odot$ $P = 2\pi\sqrt{\frac{R}{F}}$.

$$\therefore P^2 = 4\pi^2 \cdot \frac{R}{F}.$$

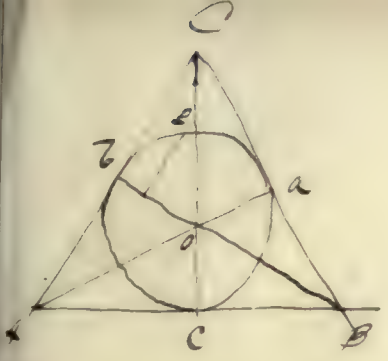
In this case R = stretched length of the string, for it equals the radius of the $\odot$ actually described

$\therefore$ substituting and making use of general notation

$$T^2 = 4\pi^2 \cdot \frac{r}{\mu(r-a)}$$

$$\text{or } T^2 \propto \frac{r}{r-a}$$

Senate House.



3 bodies repelled given masses (forces varies as dist) are fixed in the vertices of an equilateral Δ. find the velocity with which a body must be projected in direction of one of the sides from the middle point so that it may describe the inscribed ⊙

Let b be the point v the velocity of proj^n. b C. b B rectangular axes of y. x & z the posn. of the body at a time t. x. y. const^s of R. AC = a.

$$\frac{d^2x}{t} = -\mu\left(\frac{a}{2}\sqrt{3} - x\right) + 2\mu x = -\mu\left(\frac{a}{2}\sqrt{3} - 3x\right) \quad (1). \qquad \frac{d^2y}{t} = -3\mu y. \quad (2)$$

from (2) $\left(\frac{dy}{t}\right)^2 = -2\mu y^2 + C$. At the commencement $y = 0$ $\left(\frac{dy}{t}\right)^2 = v'^2$.

$$\therefore \frac{dy}{t} = \sqrt{v'^2 - 3\mu y^2} = \sqrt{3\mu}\cdot\sqrt{\frac{v'^2}{3\mu} - y^2}$$

$$\therefore t = \frac{1}{\sqrt{3\mu}}\cos^{-1}\frac{y\sqrt{3\mu}}{v'} + C.$$

$$t = 0 \quad y = 0. \quad \therefore t = \frac{1}{\sqrt{3\mu}}\left\{\cos^{-1}\frac{y\sqrt{3\mu}}{v'} - \frac{\pi}{2}\right\} \quad (3)$$

Again $\frac{d^2u}{t} + \mu\cdot\frac{a}{2}\sqrt{3} - 3\mu x = 0$ $\frac{d^2x}{t} - 3\mu x + \mu\frac{a}{2}\sqrt{3} = 0$.

$$\frac{d^2u}{t} - 3\mu u = 0 \quad \text{Let } u = e^{mt} \quad \therefore m^2 - 3\mu = 0$$

$$m^2 = 3\mu. \quad \therefore m = \pm\sqrt{3\mu}.$$

$$\therefore u = C_1 e^{t\sqrt{3\mu}} + C_2 e^{-t\sqrt{3\mu}}$$

$$\frac{d}{t}e^{-t\sqrt{3\mu}}\cdot u = -2\sqrt{3\mu}\cdot C_2 e^{-2t\sqrt{3\mu}}.$$

$$\therefore \frac{d}{t}\cdot e^{2t\sqrt{3\mu}}\cdot\frac{d}{t}e^{-t\sqrt{3\mu}}u = 0 \qquad \text{m. of the coeff}^t \text{ of } \frac{d^2u}{t} \text{ is } e^{-t\sqrt{3\mu}}$$

$$\therefore e^{-t\sqrt{3\mu}} = \frac{d}{t}\cdot e^{t\sqrt{3\mu}}\cdot\frac{d}{t}e^{-t\sqrt{3\mu}} \qquad \mathfrak{X} = -\mu\cdot\frac{a}{2}\sqrt{3}.$$

$$e^{t\sqrt{3\mu}}\cdot\frac{d}{t}\cdot e^{-t\sqrt{3\mu}}x = -\frac{\mu a}{2}\sqrt{3}\int_t e^{t\sqrt{3\mu}} = \frac{a}{2}\sqrt{\mu}\cdot e^{t\sqrt{3\mu}}.$$

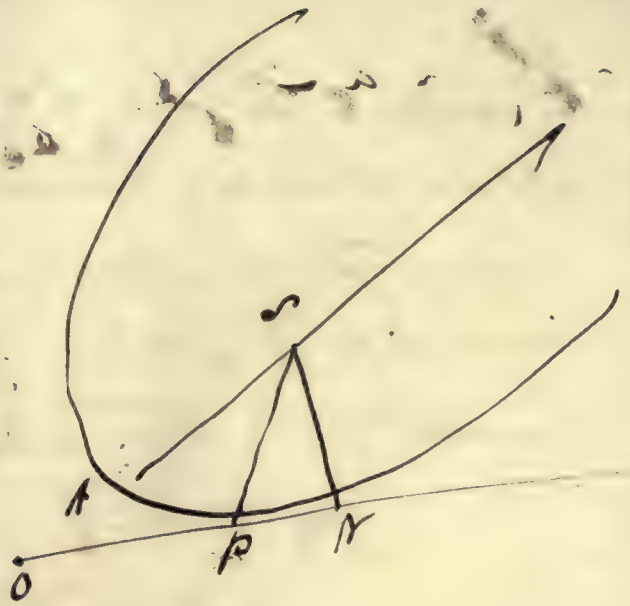
$$e^{-t\sqrt{3\mu}}x = \frac{a}{2}\sqrt{\mu}\cdot\int_t e^{-t\sqrt{3\mu}} = -\frac{a}{2\sqrt{3}}\cdot e^{-t\sqrt{3\mu}}.$$

$$\therefore x = -\frac{a}{2\sqrt{3}} + C_1 e^{t\sqrt{3\mu}} + C_2\cdot e^{-t\sqrt{3\mu}}. \qquad = \beta \text{ suppose}$$

From (3) $t\sqrt{3\mu} = \cos^{-1}\frac{y\sqrt{3\mu}}{v'} - \frac{\pi}{2}$ $\therefore \cos\left(\frac{\pi}{2} - t\sqrt{3\mu}\right)\frac{v'}{\sqrt{3\mu}} = y$

But $y^2 = \beta(a\sqrt{3}x - x^2)$ $\therefore \frac{v'^2}{3\mu}\cdot\cos^2\left(\frac{\pi}{2} - t\sqrt{3\mu}\right) = a\sqrt{3}\cdot\beta - \beta^2$

$$v'^2 = \frac{2\mu\beta(a\sqrt{3} - \beta)}{\cos^2\left(\frac{\pi}{2} - t\sqrt{3\mu}\right)}.$$



$ON = x$ $SN = y$

SP ⊥ normal to locus of S

$$d_x y = \cot \angle SPN$$

$$\therefore \text{S.P. ext } SPN = SN$$

$$SP = SN \cdot \sqrt{1 + d_x y^2}$$

But $r^2 = SN^2 = SP \cdot SA = ay\sqrt{1 + d_x y^2}$

$$1 + d_x y^2 = \frac{y^2}{a^2}$$

$$d_x y^2 = \frac{y^2 - a^2}{a^2}$$

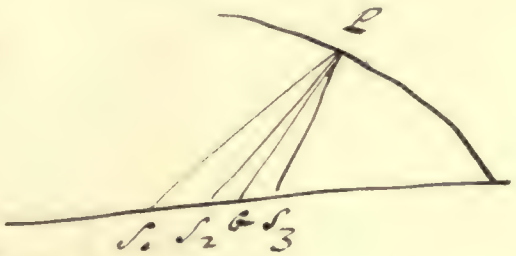
$$d_y x = \frac{a}{\sqrt{y^2 - a^2}}$$

this = ⁇ to d_x of catenary

Newton.

If $S_1, S_2 \ldots$ be centres of force, P a point in a curve such that $\phi_1(r_1) + \phi_2(r_2) + \&c.$ is constant, (where $r_1 = S_1P$ &c.) a body may be made to move uniformly in the curve if the force on P of S_1 be $d_{r_1}\phi_1(r_1)$ of S_2 be $d_{r_2}\phi_2(r_2)$ &c. If all the centres of force be situated in the same straight line, and the normal to the curve at P cut it in G the whole normal force is

$$PG.\ \frac{d_{r_1}\phi_1(r_1)}{r_1} + \frac{d_{r_2}\phi_2(r_2)}{r_2} + \&c.$$



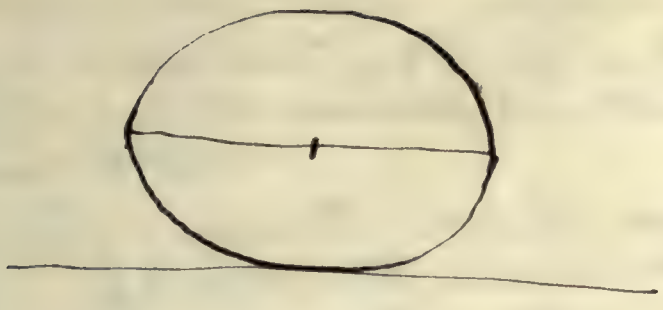
For the first law

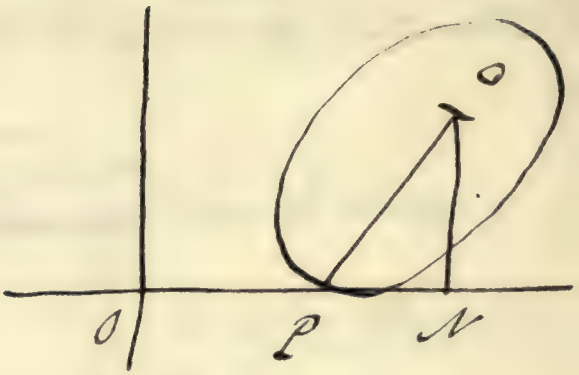
$$v^2 = c^2 - 2\int^r E.$$

$$= c^2 - 2\int\{\phi_1(r_1) + \phi_2(r_2) + \&c.\} \text{ which is const.}$$

For the second law. resolve the force to each centre in the direction of the normal and the direction $\perp$ to it. the force in direction of the normal from the first centre will be $d_{r_1}\phi(r_1).\frac{PG}{r_1}$ and similarly of the rest. $\therefore$ the whole force in direction of the normal is

$$PG\left\{\frac{d_{r_1}\phi_1(r_1)}{r_1} + \frac{d_{r_2}\phi_2(r_2)}{r_2} + \&c.\right\}.$$





$ON = x \cdot NC = y$ cont of C.
of the hei abcy which vals.

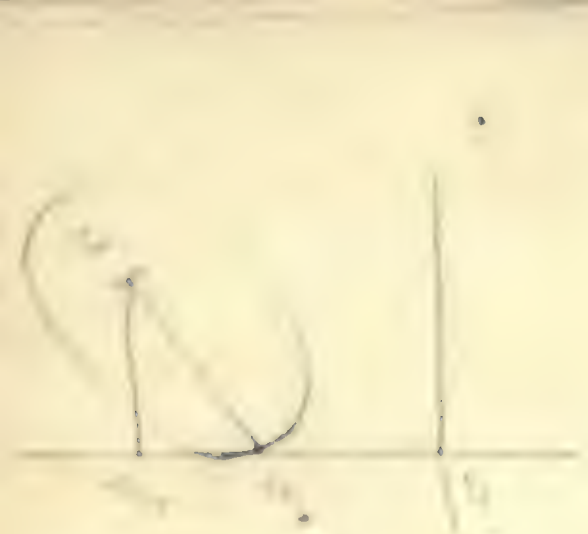
CP is a normal to the curve of C.

$\therefore CP = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

Find $CN^2 = \frac{a^2+b^2}{a^2+b^2+1}$.

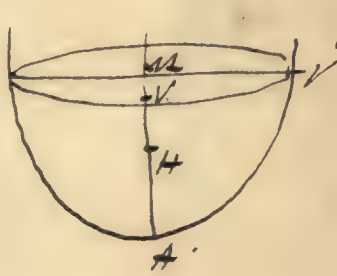
$\therefore y^2 = \frac{a^2b^2}{a^2+b^2-y^2-y^2\left(\frac{dy}{dx}\right)^2}$.

$\therefore \frac{dy}{dx} = \frac{y^2}{\sqrt{(a^2-y^2)(y^2-b^2)}}$.



A symmetrical solid generated by the revolution of a plane curve round its axis floats with its axis vertical and vertex downwards. Find its nature when the metacentre is (n) times as high as the centre of gravity of the fluid displaced.

Let $y = f(x)$ be the $= ^n$ of the generating curve.
x = depth of the axis immersed in points of $= ^{ns}$.
$\bar{x}$ = height of the cent. of grav. of the fluid displaced above the lowest point of the axis of the solid.



Then $\bar{x} = \dfrac{\pi \int x\, y^2}{\pi \int x\, y^2}$ (1).

where y = radius of the plane of flotation. $= \text{rad}^s V'$

Let H be the cent. of grav. of the fluid displaced

$\therefore HH = \dfrac{\pi . \int x\, y^2}{\pi \int x\, y^2}$ between proper limits

also moment of inertia of the plane of flotation $= \dfrac{\pi}{4} . y^4$

and of the solid immersed in the fluid $= \pi \int x\, y^2$ between proper limits

for both points of the metacentre.

$$\therefore HV = \frac{\frac{\pi}{4} y^4}{\pi \int x\, y^2} = \frac{\pi y^4}{4\pi . \int x\, y^2}$$

Therefore by the question. $HV = (n-1)\, HH$.

$$\therefore \frac{y^4}{\int x\, y^2} = 4(n-1)\, \frac{\int x\, y^2}{\int x\, y^2}$$

$$y^4 = 4(n-1) \int x\, y^2.$$

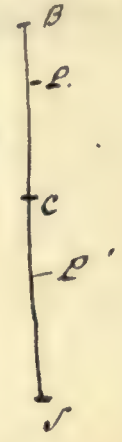
$$4 y^3 \frac{dy}{dx} = 4(n-1)\, xy^2 \text{ by differentiation.}$$

$$2y \frac{dy}{dx} = (n-1)\, 2x.$$

$$y^2 = (n-1)(x^2 - c^2)$$

is the eqn $= ^n$ to the curve.

a body falls from a point above the earth's surface equal in height to the earth's radius, to the cent. of the earth. find the whole time of motion and the velocity acqd.



Let S be the earth's centre C a pt. on its surf. L a point between C and B. $SL = x$.

Then f at C : f at L :: $SC^2 : SL^2$

$\therefore f$ at $L = \frac{a^2 g}{x^2}$.

$\therefore \frac{d^2x}{dt^2} = -\frac{a^2 g}{x^2}$

$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} + C$. when $\left(\frac{dx}{dt}\right) = 0$ $x = 2a$.

$\therefore \left(\frac{dx}{dt}\right)^2 = 2a^2 g \left\{\frac{1}{x} - \frac{1}{2a}\right\}$ $\therefore$ velocity at $C = \sqrt{2a^2 g} \cdot \sqrt{\frac{a}{2a^2}}$

$= \sqrt{ag}$.

$\therefore \frac{dt}{dx} = -\frac{1}{\sqrt{2a^2 g}} \cdot \left\{\frac{2ax}{2a - x}\right\}^{\frac{1}{2}} = -\frac{1}{\sqrt{ag}} \cdot \frac{x}{\sqrt{2ax - x^2}}$

$\therefore t = +\frac{1}{\sqrt{ag}} \left\{\sqrt{2ax - x^2} - a \operatorname{vers}^{-1} \frac{x}{a}\right\} + C$.

when $t = 0$ $x = 2a$ $\therefore 0 = +\frac{1}{\sqrt{ag}} \{\pi a\} + C$

$\therefore t = +\frac{1}{\sqrt{ag}} \left\{\sqrt{2ax - x^2} - a \operatorname{vers}^{-1} \frac{x}{a} + \pi a\right\}$.

when $x = a$ $\therefore t = +\frac{a}{\sqrt{ag}} \left\{1 + \frac{\pi}{2}\right\} = \sqrt{\frac{a}{g}} \cdot \left\{\frac{\pi}{2} + 1\right\}$.

Also f at L' : f at C :: SL' : SC $\therefore f$ at $L' = \frac{gx}{a}$.

$\therefore$ for the motion from C to S when

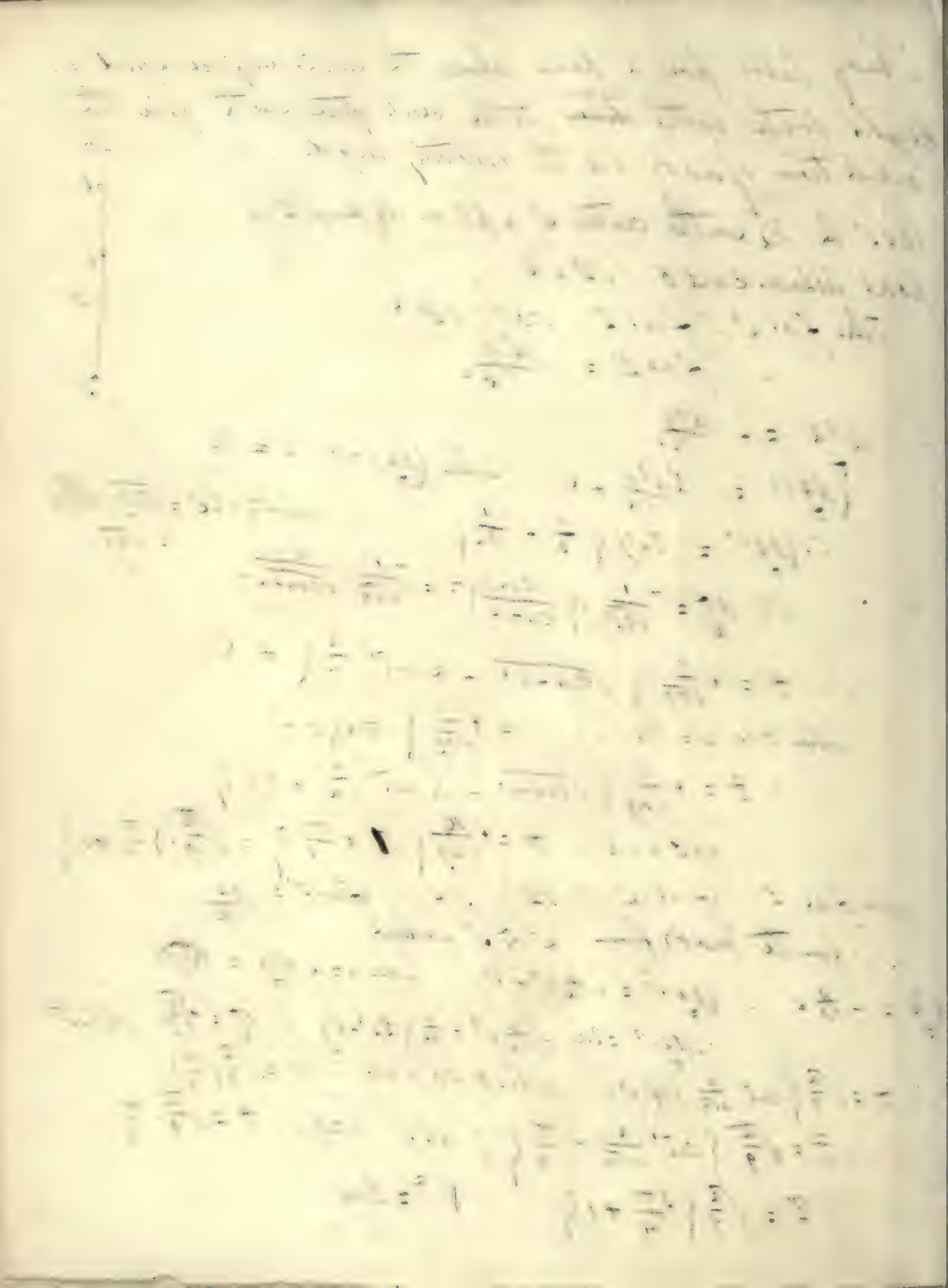
$\frac{d^2x}{dt^2} = -\frac{g}{a}x$ $\therefore \left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + C$. when $x = a$ $\frac{d^2x}{dt^2} = -(g)$

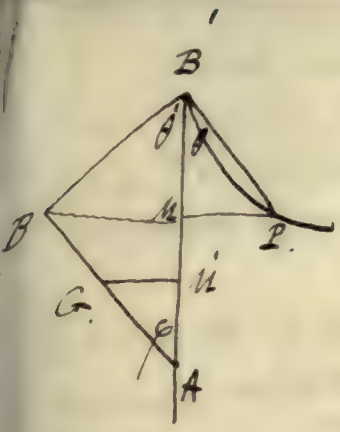
$\therefore \left(\frac{dx}{dt}\right)^2 = 2ag - \frac{g}{a}x^2 = \frac{g}{a}\{2a^2 - x^2\}$ $\therefore \frac{dt}{dx} = -\sqrt{\frac{a}{g}} \cdot \frac{-1}{\sqrt{2a^2 - x^2}}$

$\therefore t = \sqrt{\frac{a}{g}} \left\{\cos^{-1} \frac{x}{a\sqrt{2}}\right\} + C$. when $t = 0$ $x = a$ $\therefore 0 = \sqrt{\frac{a}{g}} \left\{\frac{\pi}{4}\right\} + C$

$\therefore t = \sqrt{\frac{a}{g}} \left\{\cos^{-1} \frac{x}{a\sqrt{2}} - \frac{\pi}{4}\right\}$; at S $x = 0$ $\therefore T = \sqrt{\frac{a}{g}} \cdot \frac{\pi}{4}$.

$\therefore T = \sqrt{\frac{a}{g}} \left\{\frac{3\pi}{4} + 1\right\}$. $V^2 = 2ag$.





Let the system suffer a small displacement.

Then we have $\frac{mk''}{2} + mk = C$ (1).

$$k' = B'M' = \rho' \cos\theta' + a\cos\varphi = b - a\cos\varphi.$$

$$\text{Also } \rho' \delta\theta' = 2a\,\delta\varphi.$$

$$\therefore \delta\varphi = \rho' \frac{\delta\theta'}{2a} \quad (1). \quad \text{But } b - \rho'\cos\theta' = 2a\cos\varphi$$

$$\therefore \frac{\rho'\delta\theta'}{b - \rho'\cos\theta'} = \tan\varphi.$$

$$\text{or } \tan\varphi = \frac{\rho'\delta\theta'}{\sqrt{b^2 - 2\rho' b\cos\theta' + \rho'^2}}$$

$$\text{whence } 4a^2 = b^2 - 2\rho' b\cos\theta' + \rho'^2.$$

$$\cos\theta' = \frac{b^2 + \rho'^2 - 4a^2}{2\rho' b}$$

$$a\cos\varphi = \frac{b - \rho'\cos\theta'}{2} = \left(b - \frac{b^2 + \rho'^2 - 4a^2}{2b}\right)\frac{1}{2}$$

$$= \frac{1}{4b}\left(b^2 - \rho'^2 + 4a^2\right)$$

$$\therefore k' = \frac{b^2 + \rho'^2 - 4a^2}{2b} + \frac{1}{4b}\left(b^2 - \rho'^2 + 4a^2\right)$$

$$= \frac{1}{4b}\left(3b^2 + \rho'^2 - 4a^2\right).$$

A body revolving in an ϵ^2 about a centre of force in the focus of a given point in its orbit receives an impulse in the direction of its motion find the magnitude of the impulse so that the eccentricity of the orbit may be increased by a given quantity.

Let r be the distance of the body at the time of receiving the impulse. v the original velocity, v' the velocity after the impulse. Let $a.b.$ be the semi axes of the original ellipse. $s.s'$ the space described while v becomes v' before and after receiving the impact.

Then $2a = \frac{r^2}{2r-v^2}$ $\quad 2a' = \frac{r^2}{s+s'-r} = \frac{r^2}{s'-s}$

$$b = \frac{r\sqrt{s}\sin\alpha}{\sqrt{(r-s)}} \qquad b' = \frac{r\sqrt{s's}\sin\alpha}{\sqrt{r-s'}}$$

$$\therefore e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{r^2 s \sin^2\alpha (r-s)^2}{(r-s)^2 r^4} = 1 - \frac{s(\sin^2\alpha)(r-s)}{r^2}$$

$$= \frac{r^2 - s(r-s)\sin^2\alpha}{r^2}$$

$$e'^2 = 1 - \frac{b'^2}{a'^2} \qquad = \frac{r^2 - s'(r-s')\sin^2\alpha}{r^2}$$

Let α be the quantity by which the eccentricity is to be increased

Now $e = \sqrt{1 - \frac{s(r-s)}{r^2}\sin^2\alpha} = 1 - \frac{1}{2}\frac{s(r-s)\sin^2\alpha}{r^2}$ nearly.

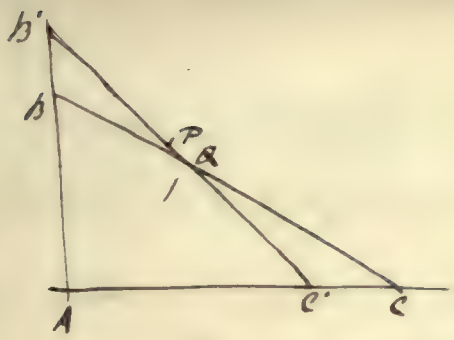
$$e' = \text{——————} = 1 - \frac{1}{2}\cdot\frac{s'(r-s')\sin^2\alpha}{r^2} \text{ nearly}$$

$$\therefore 1 - \frac{1}{2}\frac{s(r-s)\sin^2\alpha}{r^2} = 1 - \frac{1}{2}\frac{s'(r-s')\sin^2\alpha}{r^2} + \alpha.$$

$$s(r-s) = -\frac{2\alpha r^2}{\sin^2\alpha} + s'(r-s').$$

$\therefore s'$ is known

Therefore v' the velocity at P is known after the impact, wherefore $v' - v$ the increment of velocity at P is known, and consequently the magnitude of the impact.



P pt of alt^s inters^n

$\Delta AB'C' = \Delta ABC$

$ABQC'$ is common

$\therefore \Delta B'BQ = \Delta CQC'$

$\therefore B'Q \cdot QB \sin Q = CQ \cdot QC' \sin Q$

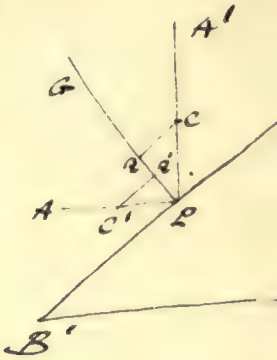
$\therefore B'Q \cdot QB = CQ \cdot QC'$ always $\therefore$ alt

When B moves up to B' in which case $BQ = Q'B + CQ = QC$

$\therefore QB^2 = CQ^2$

$\therefore QB = CQ$

A ball whose elasticity is $\frac{1}{2}$. is let fall upon a hard plane, inclined at an angle of 30° to the horizon, then the little range on the plane $= \alpha c$.



A the pt. from wh. the ball falls. $AL = s$. $GL \perp$ to the plane. CR // to the plane. $LR' = \frac{1}{2} LR$, $R'C'$ // to the plane and $= $ to RC. Then $LC'A$ is the direction of reflection

Let $\angle ALB' = (\alpha + 30^\circ)$ $\therefore \alpha =$ inclinat. of the direct^n of reflect^n to the horizon.

Now the velocity with wh. the body strikes the plane is $\sqrt{2sg}$. This is equivalent to a velocity $\frac{\sqrt{2sg}}{2}$ // to the plane and $\frac{\sqrt{6sg}}{2}$ $\perp$ to the plane. $\therefore$ the velocity of reflection will be. $\sqrt{\left(\frac{2}{4} + \frac{6}{16}\right) 2sg} = \sqrt{\frac{7}{8} sg} = \frac{1}{2}\sqrt{\frac{7}{2} sg}$.

$$\text{also } \tan C'LB' = \frac{\sqrt{6sg}}{4} \cdot \frac{2}{\sqrt{2sg}} = \frac{\sqrt{3}}{2} = \tan(\alpha + 30^\circ) \therefore \sin(\alpha + 30^\circ) = \sqrt{\frac{3}{7}}.$$

$$\text{also } \tan \alpha = \frac{\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}}}{1 + \frac{1}{2}} = \frac{1}{2\sqrt{3}} \times \frac{2}{3} = \frac{1}{3\sqrt{3}}$$

$$\therefore \cos\alpha = \frac{3\sqrt{3}}{\sqrt{28}} = \frac{3}{2}\sqrt{\frac{3}{7}}.$$

Let $R =$ required range on the inclined plane.

$$\text{Then } R = \frac{2v^2 \cos\alpha \cdot \sin(\alpha + c')}{g \cos^2 c'}$$

$$= \frac{\frac{1}{2} \cdot \frac{7}{2} sg \times \frac{3}{2}\sqrt{\frac{3}{7}} \times \sqrt{\frac{3}{7}}}{g \cdot \frac{3}{4}} = s \times \frac{3}{2}.$$

$\therefore$ range on the plane $= \frac{3}{2}$ height from which the body descended.

$$y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha}$$

$$a = b\tan\alpha - \frac{g}{2v^2\cos^2\alpha}\cdot b^2 \quad \&c$$

If a projectile passes thro' 3 pts. ab; $a'b'$; $a''b''$; then &c.

$$\frac{ab''-a''b}{a''(a''-a)} = \frac{bb''\tan\alpha - \frac{g}{2v^2\cos^2\alpha}b^2b'' - b''b\tan\alpha + \frac{g}{2v^2\cos^2\alpha}b''^2b}{a''\left(b''\tan\alpha - \frac{g}{2v^2\cos^2\alpha}b''^2 - b\tan\alpha + b^2\frac{g}{2v^2\cos^2\alpha}\right)}$$

$$= \frac{-bb''(b-b'')\frac{g}{2v^2\cos^2\alpha}}{a''\left((b''-b)\tan\alpha + (b-b'')(b+b'')\frac{g}{2v^2\cos^2\alpha}\right)}$$

$$= \frac{-bb''\frac{g}{2v^2\cos^2\alpha}}{b''\tan\alpha - \frac{g}{2v^2\cos^2\alpha}b''^2\left((b+b'')\frac{g}{2v^2\cos^2\alpha} - \tan\alpha\right)}$$

similarly $$\frac{ab'-a'b}{a'(a'-a)} = \frac{-bb'\cdot\frac{g}{2v^2\cos^2\alpha}}{b'\tan\alpha - \frac{g}{2v^2\cos^2\alpha}b'^2\left((b+b')\frac{g}{2v^2\cos^2\alpha} - \tan\alpha\right)}$$

$$\therefore \frac{ab'-a'b}{a'(a'-a)} > = < \frac{ab''-a''b}{a''(a''-a)}$$

$$\frac{b'}{b'\tan\alpha - \frac{g}{2v^2\cos^2\alpha}b'^2\left((b+b')\frac{g}{2v^2\cos^2\alpha} - \tan\alpha\right)} > = < \frac{b''}{\left(b''\tan\alpha - \frac{g}{2v^2\cos^2\alpha}b''^2\right)\left((b+b'')\frac{g}{2v^2\cos^2\alpha} - \tan\alpha\right)}$$

$$b'b''\tan\alpha - \frac{g}{2v^2\cos\alpha}\times b'b''\left((b+b'')\frac{g}{2v^2\cos^2\alpha} - \tan\alpha\right) > = < \left(b'b''\tan\alpha - \frac{g}{2v^2\cos^2\alpha}b'b''\right)\left((b+b'')\frac{g}{2v^2\cos^2\alpha} - \tan\alpha\right)$$

$$(b+b'')\frac{g}{2v^2\cos^2\alpha}\cdot\tan\alpha - (bb''+b''^2)\frac{g^2}{4v^4\cos^4\alpha} - \tan^2\alpha + \frac{g}{2v^2\cos^2\alpha}b''\tan\alpha > = < \times \text{ —}$$

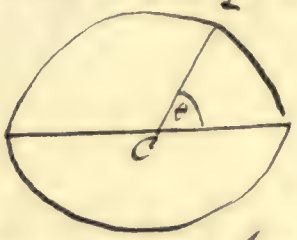
$$(b+b'')\frac{g}{2v^2\cos^2\alpha} \quad \text{— — — — — —}$$

But the latter is equal.

$$\therefore \frac{ab'-a'b}{a'(a'-a)} = \frac{ab''-a''b}{a''(a''-a)}.$$

A body revolves in an ℓ. ellipse force in centre. Show that its distance from centre after completing $\frac{1}{8}$ part of the revolution $= \sqrt{\frac{a^2+b^2}{2}}$ measuring from the ends of major axis

i.e. the area swept out by the rad. vector is ∴ ℓ to the times of describing the. the radius vector must have swept out $\frac{1}{8}$ part of the ℓ after the proposed time ∴ sectorial area $LCA = \frac{\pi ab}{8}$ when $CL = r$



Now area of the sector =

$\int_0^{\theta} \int_0^r r$ between proper limits

$= \frac{1}{2}\int_0^{\theta} \frac{b^2}{1-e^2\cos^2\theta} = \frac{b^2}{2\sqrt{1-e^2}} \tan^{-1}\frac{\tan\theta}{\sqrt{1-e^2}}$ for $A = 0$ when $\theta = 0$

$\therefore \frac{b^2}{2\sqrt{1-e^2}} \tan^{-1}\frac{\tan\theta}{\sqrt{1-e^2}} = \frac{\pi b^2}{8\sqrt{1-e^2}}$ or $\tan^{-1}\frac{\tan\theta}{\sqrt{1-e^2}} = \frac{\pi}{4}$

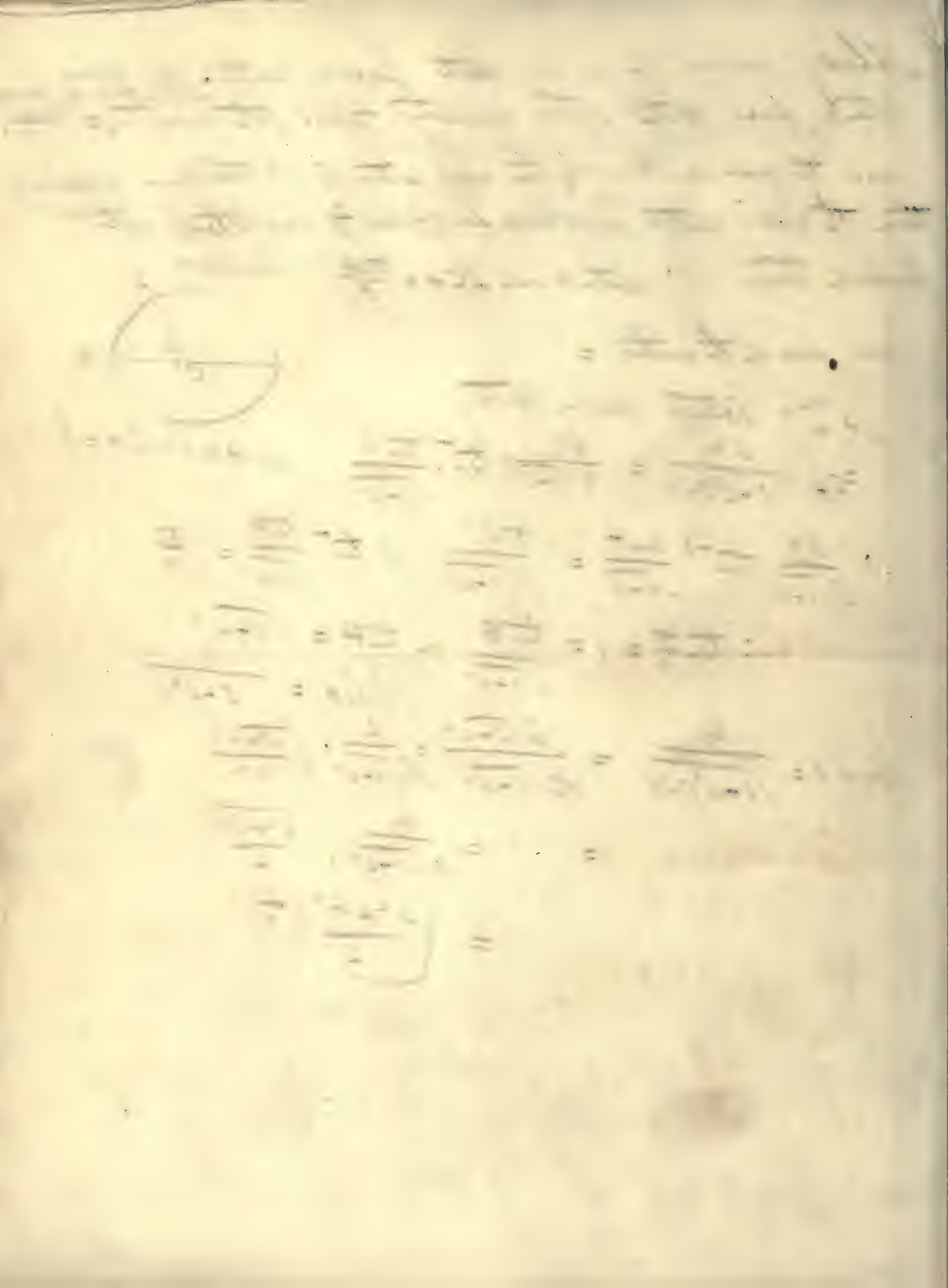
$\therefore \tan\frac{\pi}{4} = 1 = \frac{\tan\theta}{\sqrt{1-e^2}}$ or $\tan\theta = \sqrt{1-e^2}$

$\therefore \cos^2\theta = \frac{1}{2-e^2}$

But $r = \frac{b}{\sqrt{1-e^2\cos^2\theta}} = \frac{b\sqrt{2-e^2}}{\sqrt{2}\sqrt{1-e^2}} = \frac{b}{\sqrt{2}\sqrt{1-e^2}} \cdot \sqrt{\frac{a^2+b^2}{a^2}}$

$= = \frac{b}{a\sqrt{1-e^2}} \sqrt{\frac{a^2+b^2}{2}}$

$= \left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}}$



$x^2+y^2 = (a+lx+my)^2 = a^2+2alx+l^2x^2+m^2y^2+2amy+2lmxy.$

$\therefore (1-l^2)x^2+(1-m^2)y^2-2lmxy-2alx-2amy-a^2=0$

Change the origin of co-ords by changing x into $x'+h$, y into $y'+k$

$(1-l^2)\{x'^2+2x'h+h^2\}+(1-m^2)\{y'^2+2y'k+k^2\}-2lm\{x'y'+x'k+y'h+hk\}$
$-2al(x'+h)-2am(y'+k)-a^2=0$

$(1-l^2)x'^2+(1-m^2)y'^2-2lmx'y'+\{2h(1-l^2)-2lmk-2al\}x'$
$+\{2k(1-m^2)-2lmh-2am\}y'+(1-l^2)h^2+(1-m^2)k^2-2lmhk+\ldots=$

Equate to zero the coeffts of x' and y'

$(1-l^2)h-lmk-al=0$

$(1-m^2)k-lmh-am=0$

$\therefore k=\dfrac{lmh+al}{1-l^2}=\dfrac{(1-l^2)h-al}{lm}$ $\therefore l^2mh+alm^2=(1-l^2-m^2+l^2m^2)h$
$-al+al^2m^2$

or $h=\dfrac{al}{1-l^2-m^2}$, $k=\dfrac{am}{1-l^2-m^2}$

$\therefore$ suppressing the accents we have

$(1-l^2)x^2+(1-m^2)y^2-2lmxy+(1-l^2)h^2+(1-m^2)k^2-2lmhk-2alh-2amk-a^2$

Change the direction of the axes by changing
x into $x'\cos\theta-y'\sin\theta$, y into $x'\sin\theta+y'\cos\theta$.

$$(1-m^2)\{x'^2\cos^2\theta - 2x'y'\cos\theta\sin\theta + y'^2\sin^2\theta\}$$

$$(1-n^2)\{x'^2\sin^2\theta + 2x'y'\cos\theta\sin\theta \mp y'^2\cos^2\theta\}$$

$$-2mn\{x'^2\sin\theta\cos\theta + x'y'\{\cos^2\theta - \sin^2\theta\} + y'^2\sin\theta\cos\theta\}$$

$$\pm(1\mp m^2)h^2 + (1-n^2)k^2 - 2mnhk - 2lmh - 2lnk - a^2 = 0$$

$$\mp\{(1-m^2)\cos^2\theta + (1-m^2)\sin^2\theta - 2mn\sin\theta\cos\theta\}x'^2 = Ax'^2$$

$$+\{(1-n^2)\sin^2\theta + (1-n^2)\cos^2\theta + 2mn\sin\theta\cos\theta\}y'^2 = By'^2$$

$$2x'y'\{(1-m^2)\cos\theta\sin\theta - (1-n^2)\cos\theta\sin\theta + mn\{\cos^2\theta - \sin^2\theta\}\}$$

$$+(1-m^2)h^2 + (1-n^2)k^2 - 2mnhk - 2lmh - 2lnk - a^2 = 0$$

Equate to zero the coeff. of $x'y'$ and we have

$$(1-m^2) - (1-n^2)\sin 2\theta + mn\cos 2\theta = 0$$

$$(1-m^2) - (1-n^2)\tan 2\theta = -mn \qquad \therefore \tan 2\theta = -\frac{mn}{n^2-m^2} = \frac{mn}{m^2-n^2}$$

$$\cos 2\theta = \frac{1}{\sqrt{1+\tan^2 2\theta}} = \frac{1}{\sqrt{1+\frac{m^2n^2}{m^4-2m^2n^2+n^4}}} = \frac{m^2-n^2}{\sqrt{m^4-m^2n^2+n^4}}$$

$$\sin 2\theta = \frac{\tan 2\theta}{\sqrt{1+\tan^2 2\theta}} = \frac{\frac{mn}{m^2-n^2}}{\sqrt{1+\tan^2 2\theta}} = \frac{mn}{\sqrt{m^4-m^2n^2+n^4}}$$

Add together A and B and we have

$$1-m^2+1-n^2 = 2-m^2-n^2 = A+B$$

$$(1-m^2)\cos 2\theta - (1-n^2)\cos 2\theta + 2mn\sin 2\theta = A-B$$

$$(n^2-m^2)\cos 2\theta \mp 2mn\sin 2\theta = A-B$$

Substitute the values of $\cos 2\theta$, $\sin 2\theta$,

$$\frac{(k^2-m^2)(m^2-n^2)}{\sqrt{m^4-m^2n^2+n^4}} + \frac{2mn}{\sqrt{m^4-m^2n^2+n^4}} - \frac{m^4+n^4}{\sqrt{m^4-m^2n^2+n^4}}$$

$\therefore A+B = 2-m-n^2$

$-A+B = \left\{ \frac{m^4+n^4}{\sqrt{m^4-m^2n^2+n^4}} \right\}$

$A = \frac{1}{2}\left\{ 2-m^2-n^2 - \frac{m^4+n^4}{\sqrt{m^4-m^2n^2+n^4}} \right\}$

$B = \frac{1}{2}\left\{ 2-m^2-n^2 + \frac{m^4+n^4}{\sqrt{m^4-m^2n^2+n^4}} \right\}$.

$\therefore$ the eqn becomes. suppressing the accents.

$$\frac{1}{2}\left\{ 2-m^2-n^2 - \frac{m^4+n^4}{\sqrt{m^4-m^2n^2+n^4}} \right\} x^2 + \frac{1}{2}\left\{ 2-m^2-n^2 + \frac{m^4+n^4}{\sqrt{m^4-m^2n^2+n^4}} \right\} y^2$$

$$+ (1-m^2)\frac{a^2m^2}{(1-m^2-n^2)^2} + (1-n^2)\frac{a^2n^2}{(1-m^2-n^2)^2} - 2mn\left\{ \frac{a^2mn}{(1-m^2+n^2)^2} \right\}.$$

$$- \frac{2a^2m^2}{1-m^2-n^2} - \frac{2a^2n^2}{1-m^2-n^2} - a^2 = 0$$

$$= \frac{1}{2}\left\{ 2-m^2-n^2 + \frac{m^4+n^4}{\sqrt{m^4-m^2n^2+n^4}} \right\} y^2 + \frac{1}{2}\left\{ 2-m^2-n^2 - \frac{m^4+n^4}{\sqrt{m^4-m^2n^2+n^4}} \right\} x^2$$

$$+ \frac{a^2}{(1-m^2-n^2)^2} = 0$$

$\therefore$ If the coeffts of x^2 & y^2 are negative the curve is an ⊖

Trigonometrical Problem.

The angles of a plane triangle form a geom. progress. whose comm. rat. is $\frac{1}{2}$ shew that the greatest side =

$$2 \text{ perim.} \left\{ \sin 12^\circ\ 57'\ 25'' \tfrac{5}{7} \right\}$$

Let p = perimeter

$a.b.c$ = sides opposite $\theta, \frac{\theta}{2}, \frac{\theta}{4}$

Then $p = a+b+c = a\left\{ \dfrac{\sin\theta + \sin\frac{\theta}{2} + \sin\frac{\theta}{4}}{\sin\theta} \right\}$

But $\sin\theta + \sin\frac{\theta}{2} + \sin\frac{\theta}{4} = 2\left\{ \cos\frac{\theta}{2}\cos\left(\frac{\theta}{4}+\frac{\theta}{8}\right) + \cos\frac{\theta}{4}\cos\left(\frac{\theta}{2}+\frac{\theta}{8}\right)\right\} + \sin\frac{\theta}{4}$

$= 4\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8} - 2\sin\frac{\theta}{8}\sin\left(\frac{\theta}{2}+\frac{\theta}{4}\right) + \sin\frac{\theta}{4}$

$= 4\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}$ since $\sin\left(\frac{\theta}{2}+\frac{\theta}{4}\right) = \cos\frac{\theta}{8}$

$\therefore p = a\ \dfrac{4\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}}{8\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\sin\frac{\theta}{8}}$

or $a = 2p\sin\frac{\theta}{8} = 2p\sin 12^\circ\ 57'\ 28'' \tfrac{5}{7}$

When an Δ is inscribed in a semicircle with its axis major $\parallel$ to the diam. if $a.b$ be the semi-axes of the Δ and r the rad of the $\odot$, $a^4 = r^2(a^2-b^2)$
Hence shew that the area of the greatest $\Delta = \frac{2\pi r^2}{3\sqrt{3}}$, & the vol of the greatest spheroid $= \pi r^3 \frac{\sqrt{3}}{4}$.

Let $x.y$ be the co-ord. of one of the pts of contact of the Δ and $\odot$, the centre of the circle being taken as origin

Then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2y}{b}$ ——— (1)

& $x^2 + y^2 = r^2$ ——— (2)

eliminating x we have

$y^2\left(\frac{1}{b^2} + \frac{1}{a^2}\right) = \frac{2y}{b} - \frac{r^2}{a^2}$, or $y^2 - \frac{2a^2b}{a^2-b^2}y + \frac{r^2b^2}{a^2-b^2} = 0$

and since the values of y from this eqn. must be equal we have

$\frac{a^4b^2}{(a^2-b^2)^2} = \frac{r^2b^2}{a^2-b^2}$ or $a^4 = r^2(a^2-b^2)$ ——(1)

Now the area of the Δ $\propto ab$ $\therefore$ diff. with respect to a

$b + a\frac{db}{da} = 0$; $a - b\frac{db}{da} = \frac{2a^3}{r^2}$ (from 1)

$\therefore a^2 + b^2 = \frac{2a^4}{r^2} = 2(a^2-b^2)$ from (1). $\therefore a^2 = 3b^2$

and $a^2 = r^2\left(1-\frac{1}{3}\right) = \frac{2}{3}r^2$, area $= \pi ab = \frac{\pi a^2}{\sqrt{3}} = \frac{2\pi r^2}{3\sqrt{3}}$ ---(2)

Again vol of generating spheroid $= \frac{4}{3}\pi a^2 b$, diff.

$2ab + a^2\frac{db}{da} = 0$, and as before $a - b\frac{db}{da} = \frac{2a^3}{r^2}$

$\therefore a(a^2+2b^2) = \frac{2a^5}{r^2}$, or $a^2+2b^2 = \frac{2a^4}{r^2} = 2(a^2-b^2)$

$\therefore a^2 = 4b^2$ or $a = 2b$

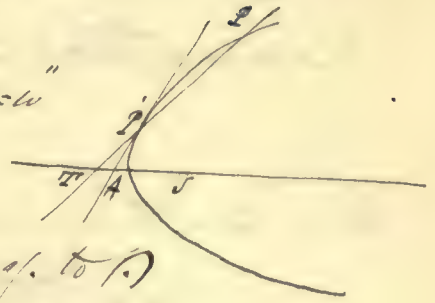
$\therefore a^2 = r^2\left(1-\frac{1}{4}\right) = \frac{3}{4}r^2$

vol $= \frac{4}{3}\pi ab = \frac{2}{3}\pi a^3 = \frac{\pi r^3\sqrt{3}}{4}$

To find the Polar eqn. to the Tangent of a ⊃

A the vertex, S the focus. $P'P''$ a secant.

$\angle ASP' = \theta'$, $\angle ASP'' = \theta''$, $\angle TP'S = \omega'$, $\angle TP''S = \omega''$



Then $\frac{\sin \omega'}{\sin \omega''} = \frac{SP''}{SP'} = \frac{1+\cos\theta'}{1+\cos\theta''}$ by the eqn. to ⊃

$\& \omega' - \omega'' = \theta'' - \theta'$

$$\therefore \frac{\sin \omega'}{2 \sin \frac{\omega' - \omega''}{2} \cos \frac{\omega' - \omega''}{2}} = \frac{1+\cos\theta'}{2 \sin \frac{\theta'' - \theta'}{2} \sin \frac{\theta'' + \theta'}{2}}$$

$$\text{or} \quad \frac{\sin \omega'}{\cos \frac{\omega' + \omega''}{2}} = \frac{1+\cos\theta'}{\sin \frac{\theta'' + \theta'}{2}}$$

Now let the secant become a Tangent at the pt. $r'\theta'$

Then $\theta'' = \theta'$, $\omega'' = \omega'$

$$\therefore \tan \omega' = \frac{1+\cos\theta'}{\sin\theta'} = \tan\left(\frac{\pi}{2} - \frac{\theta'}{2}\right)$$

$$\omega' = \frac{\pi}{2} - \frac{\theta'}{2}$$

Let now $r.\theta.$ be the co-ords of any point in the tangent at P'. Then $\because r' = \frac{m}{\cos^2 \frac{\theta'}{2}}$ by the eqn. to the ⊃ we have

$r' \sin(\omega' + \theta + \theta') = r' \sin \omega$, or $r \cos\left(\theta - \frac{\theta'}{2}\right) = r' \cos \frac{\theta'}{2}$

$$\text{or } r = m \sec \frac{\theta'}{2} \sec\left(\theta - \frac{\theta'}{2}\right).$$

2

In the axis of a ⊃ infin a point such that if any Chd. be drawn through it, the angle subtended at the vertex may be a right angle.

The eqn to the lines joining the vertex with the pts of intersn of ⊃ with the ⊃ are included in the eqn

$$(y - ax)(ay + x) = 0$$

$$\text{or } y^2 - x^2 + \left(\frac{1}{a} - a\right)xy = 0$$

Combining this = n with that to the ⊃

$$y^2 = 4mx$$

we get (dividing by x)

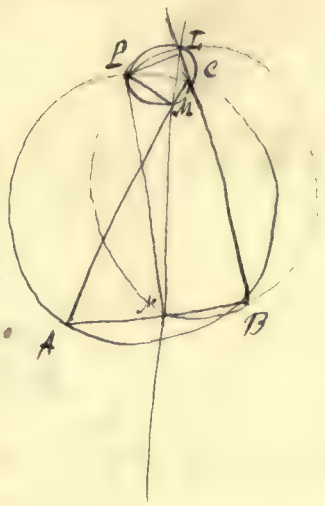
$$4m - x + \left(\frac{1}{a} - a\right)y = 0$$

an = n which being true for both points of contact is that to the right line which joins them;

Making $y = 0$ we have for the point where it cuts the axis $x = 4m$

which is independt of a; and therefore any Chd drawn through this point has the required property ——

The feet of the ⊥s from any pt in a ⊙ on the sides of the inscribed △ lie in a st. line.



Let PI, PM, PN be the ⊥s
from P on the sides of the △ ABC

Describe ⊙ through the pts PICM,
PIBN.

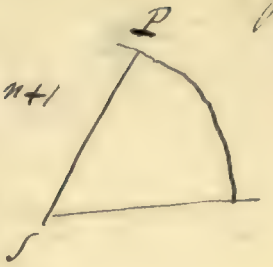
Then ∠ PIM = PCM, = PBN

∴ ∠ PIN = PBN

and since ∠s PIM, PIN are equal

IM and IN are in the same straight line.

7. Find the law of force. Prob 4.



suppose $\frac{d^2s}{dt^2} = \mu r^n$ $\therefore \left(\frac{ds}{dt}\right)^2 = \frac{2\mu}{(n+1)} r^{n+1}$

$\therefore v^2 \propto r^{n+1}$.

velocity in a $\odot \propto \frac{h}{p}$

$\propto \frac{h}{r\sin\theta}$ ~~$\sin\theta = \angle SPZ$~~

$\propto \frac{h}{r}$ $\therefore \propto \frac{1}{r}$

$\therefore$ (velocity)^2 in a curve $\propto \frac{1}{r^2}$

But the velocity in the proposed curve has a constant ratio to the velocity in a circle

$\therefore r^{n+1} = r^{-2}$ or $n+1 = -2$

$n = -3$

$\therefore F \propto r^{-3}$

$\frac{v}{v_1} = \frac{1}{r_2}$

$= \frac{1}{2} \mu r$

$= \frac{1}{2} \mu r^{n+1}$

therefore $\mu u^3 = h^2 u^2 \left(\frac{d^2u}{d\theta^2} + u\right)$

or $\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} u$ $\quad v^2 = \frac{1}{2} v_1^2$

$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta} = \frac{\mu}{h^2} 2u\frac{du}{d\theta}$

$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu}{h^2} u^2 + C$

$h = VR\sin\theta$ $\quad r\sin\theta = \pm h$

$u\sin\theta = \pm h$ $\quad \therefore \left(\frac{du}{d\theta}\right)^2 = \frac{R^2\cos^2\theta}{R^2}$.

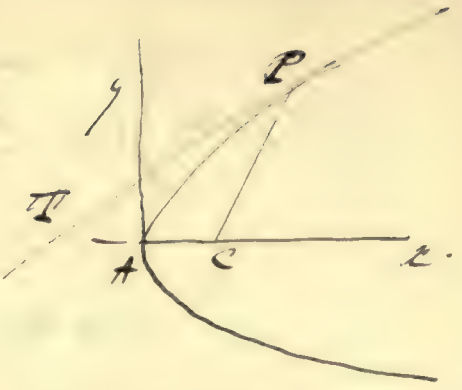
$\therefore \frac{\cos^2\theta}{R^2} + \frac{1}{h^2} - \frac{\mu}{h^2 R^2} = C$

$\frac{1}{R^2\sin^2\theta} + \left(\frac{h^2 - u}{h^2}\right)\frac{1}{R^2} = C = \frac{V^2R^2 + h^2 - \mu}{h^2R^2}$

$\therefore \left(\frac{du}{d\theta}\right)^2 = \frac{V^2R^2 + h^2 - \mu}{R^2h^2} - \left(\frac{h^2 - \mu}{h^2}\right)u^2$ $\therefore \theta =$

31

+ the origin of the central force.



$CL = r$. $\angle ACL = \theta$

$r = \frac{a}{\cos^2\theta}$ $\quad \frac{2a}{1+\cos\theta}$

~~B~~ $u = \cos^2\theta$

$\therefore u = \frac{1+\cos\theta}{2a}$.

$\frac{d^2u}{d\theta^2} + u = \frac{1}{2a}$. $\quad F = \frac{h^2}{r^2}\left\{\frac{d^2u}{d\theta^2} + u\right\} = \frac{h^2}{2a}\frac{1}{r^2} = \frac{\mu}{r^2}$

$\therefore$ the law of force is that of r^{-2}.

$\therefore \mu = \frac{h^2}{2a}$ is the absolute force at L.

Let this be doubled then $\frac{h^2}{a}$ = absolute force at L.

$\therefore F = h^2u^2\left\{\frac{d^2u}{d\theta^2} + u\right\} = \frac{h^2}{a}\frac{1}{r^2}$

$\frac{d^2u}{d\theta^2} + u = \frac{1}{a}$.

$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta} = \frac{2}{a}\frac{du}{d\theta}$

$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2}{a}u + C$

Let $\frac{1}{r} = \frac{1}{R}$

$\frac{1}{R^2\sin^2\alpha} + -\frac{2}{aR} = C'$ where $\alpha = \angle CLT$.

$\frac{a - 2R\sin^2\alpha}{aR^2\sin^2\alpha} = C'$

$\therefore \left(\frac{du}{d\theta}\right)^2 = \frac{a-2R\sin^2\alpha}{R^2\sin^2\alpha\, a} + \frac{1}{a^2} - \left(\frac{1}{a} - u\right)^2$

$\left(\theta + C'\right) = \cos^{-1}\frac{\frac{1}{a} - u}{\sqrt{\frac{a-2R\sin^2\alpha}{aR^2\sin^2\alpha} + \frac{1}{a^2}}}$

$$\therefore \frac{1}{r} = \frac{1}{a} \pm \sqrt{\frac{a - 2R\sin^2\alpha}{a \cdot R^2 \sin^2\alpha} + \frac{1}{a^2}} \cos(\theta + C').$$

The + sign taken (i) for the form

$$\frac{1}{r} = \frac{a(1-e^2)}{} \quad \frac{1 + e\cos(\theta + C')}{a(1-e^2)}.$$

The two signs are the same $\therefore$ the body will describe an ellipse about C.

$$u = \frac{1}{a}\left(\cos\frac{\theta}{n}\right)^n$$

$$\frac{d_\theta u}{u} = -\tan\frac{\theta}{n} \qquad d_\theta u = -u\tan\frac{\theta}{n}$$

$$\therefore d_\theta^2 u = -\frac{u}{n}\sec^2\frac{\theta}{n} - d_\theta u \tan\frac{\theta}{n}$$

$$= -\frac{u}{n}\sec^2\frac{\theta}{n} + u\left(\sec^2\frac{\theta}{n} - 1\right)$$

$$d_\theta^2 u + u = \left(1 - \frac{1}{n}\right) u \sec^2\frac{\theta}{n}$$

$$= \frac{n-1}{n}\, u \cdot \left(\frac{1}{ua}\right)^{\frac{2}{n}}$$

$$= \frac{n-1}{n} \cdot \frac{u}{a^{\frac{2}{n}} u^{3-\frac{2}{n}}}$$

$$\therefore P = \frac{h^2}{a^{\frac{2}{n}}} \cdot \frac{n-1}{n} \cdot u$$

$a^2 = \frac{1}{a^2} \sec 2\theta.$

$2\log u = \log \frac{1}{a^2} + \log \sec 2\theta$

$\frac{du}{u} = \frac{\sec 2\theta \tan 2\theta}{\sec 2\theta} = \tan 2\theta.$

$\frac{du}{d\theta} = u \tan 2\theta.$

$\frac{d^2u}{d\theta^2} = 2u \sec^2 2\theta + \tan 2\theta \frac{du}{d\theta}$

$= 2u \sec^2 2\theta + u \tan^2 2\theta$

$= 3u \sec^2 2\theta - u$

$\therefore \frac{d^2u}{d\theta^2} + u = 3u \sec^2 2\theta.$

$= \frac{3}{a} \sec^2 2\theta = 3a^4 \frac{1}{a^5 (\cos 2\theta)^{\frac{5}{2}}} = \frac{3a^4}{r^5}$

$\therefore t = h^2 u \left\{ \frac{d^2u}{d\theta^2} + u \right\} = \frac{3a^4 h^2}{r^7} = \frac{\mu}{r^7} \therefore \propto r^{-7}.$

also $\frac{dt}{d\theta} = \frac{r^2}{h} \therefore t = \frac{1}{h} \int r^2 = \frac{a^2}{h} \int \cos 2\theta$

$= \frac{a^2}{h^2} \sin 2\theta + C.$

$= \frac{a^2}{h}$ between the

limits $+45^\circ$ and -45°

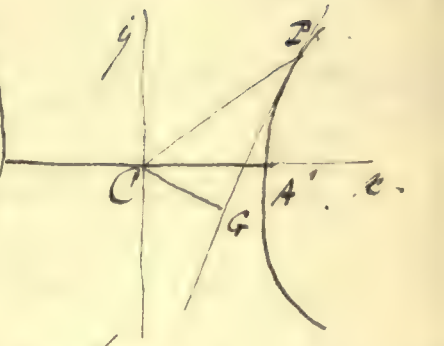
But $3a^4 h^2 = \mu \therefore h = \frac{\sqrt{\mu}}{a^2 \sqrt{3}}$

$\therefore t = \frac{a^4 \sqrt{3}}{\sqrt{\mu}}.$

i.e. the law of force by w.h. a particle may be made to desc. the lemniscate
& cent. of force coincides with the node. Hence the time of describing one
of the ovals.

$\frac{1}{p^2} = \frac{x^2 - a^2 + b^2}{a^2 b^2}$ where $p = \perp$ on the tangent

$F = -\frac{h^2}{2} \frac{d}{dr}\left(\frac{1}{p^2}\right) = -\frac{h^2}{2} \cdot \frac{2r}{r^4} = -\mu r.$



Hence the law of force is Kah after discovered.

Let θ be the ang. $CP = r$ $\angle PCA' = \theta$.

Then we have generally $\frac{d\theta}{dt} = \frac{h}{p^2} \therefore \frac{dt}{d\theta} = \frac{r^2}{h}$

But $r^2 = \frac{a^2}{-1 + e^2 \cos^2\theta} = \frac{b^2}{(1 - e^2)\cos^2\theta + \sin^2\theta}$

$= \frac{b^2 \sec^2\theta}{-1 + e^2 + \tan^2\theta}.$

$\therefore \frac{dt}{d\theta} = \frac{b^2 \sec^2\theta}{h\,(-1 + e^2 + \tan^2\theta)}$

$\therefore t = \frac{b^2}{h} \cdot \frac{1}{\sqrt{1 - e^2}} \tan^{-1}\left(\frac{\tan\theta}{\sqrt{1 - e^2}}\right)$

where t = time of describing $\angle PCA$.

$t = \frac{b^2}{h} \tan^{-1} \frac{\tan\theta}{\sqrt{2}}$

$= \frac{b^2}{h}\,\theta$

Central Forces.

1. Find the law of force by wh a particle may be made to desc. the lemniscate whose $=^n$ is $r^2 = a^2 \cos 2\theta$, the c^r of force coinciding with the node, & give the time of desc^g one of the ovals.

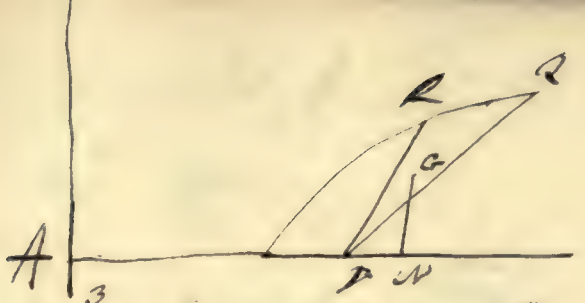
2. A particle desc. an = iat. $\propto$ round a c^r of force attracted as the c^r; find the law of force and the $\angle$ wh the particle will desc. about the c^r from the apse in a g^n time.

3. A particle is revolving ~~about~~ in a) about a c^r of force in the foci, & when it arrives at a g^n dist. from the foci the attr. force is suddenly doubled; det. the nature of the subsequent path of the particle.

4. A particle moves in a spiral, such the $=^n$ is $r = a\left(\sec\frac{\theta}{n}\right)^n$, about a c^r of force in the pole; find the law of force.

5. If a force $\propto$ as $\frac{1}{\text{dist}^3}$, & a body be projected from an apse with a vel. wh is $\sqrt{\frac{2}{3}}$ the vel. in a $\odot$ at the same dist. as $1:\sqrt{3}$, find the $=^n$ of the curve described.

$$\bar{k} = \frac{2}{3\sigma} \frac{\int_0 r^3 \cos\theta}{\int_0 r^2}$$



$$r = \frac{a}{1-\cos\theta} = \frac{a}{2\sin^2\frac{\theta}{2}} \quad \therefore r^3 \cos\theta = \frac{a^3 \cos\theta}{8\sin^6\frac{\theta}{2}} = a^3 \left\{ \frac{\cos^2\frac{\theta}{2}}{8\sin^6\frac{\theta}{2}} \right.$$

$$= a^3 \left\{ \frac{1 - 2\sin^2\frac{\theta}{2}}{8\sin^6\frac{\theta}{2}} \right\} = a^3 \left\{ \frac{1}{8\sin^6\frac{\theta}{2}} - \frac{2}{8\sin^4\frac{\theta}{2}} \right.$$

$$\int \frac{1}{8\sin^6\frac{\theta}{2}} = \int_0 \frac{\frac{1}{\cos^6\frac{\theta}{2}}}{8\sin^6\frac{\theta}{2}} = \int_0 \frac{\sec^2\frac{\theta}{2}}{8\tan^6\frac{\theta}{2}\left(\frac{1}{1+\tan^2\frac{\theta}{2}}\right)^2} = \int_0 \frac{1+3\tan^2\frac{\theta}{2}+\tan^4\frac{\theta}{2}}{8\tan^6\frac{\theta}{2}} \, d\tan\frac{\theta}{2}$$

$$= -\frac{1}{5}\tan^{-5}\frac{\theta}{2} - \frac{2}{3}\tan^{-3}\frac{\theta}{2} - \tan\frac{\theta}{2}.$$

$$\int -\frac{2}{8\sin^4\frac{\theta}{2}} = \int_0 -\frac{2\sec^2\frac{\theta}{2}\left(1+\tan^2\frac{\theta}{2}\right)}{8\tan^4\frac{\theta}{2}} = +\frac{2}{3}\tan^{-3}\frac{\theta}{2} + 2\tan\frac{\theta}{2}.$$

$$\int \frac{1}{8\sin^6\frac{\theta}{2}} + \frac{1}{8\sin^4\frac{\theta}{2}} = -\frac{1}{5}\tan^{-5}\frac{\theta}{2} - \tan^{-3}\frac{\theta}{2} - 2\tan\frac{\theta}{2}.$$

$$\therefore a^3 \int_0 \left(\frac{1}{8\sin^6\frac{\theta}{2}} - 2\frac{1}{8\sin^4\frac{\theta}{2}} \right) = a^3 \left\{ +\frac{1}{5}\tan^{-5}\frac{\theta}{2} + \tan^{-1}\frac{\theta}{2} \right.$$

$$\int r^3 = \int_0 \frac{a^2}{8\sin^4\frac{\theta}{2}} = a^3 \int_0 \frac{1}{8\sin^4\frac{\theta}{2}} = a^3 \left\{ +\frac{1}{3}\tan^{-3}\frac{\theta}{2} + \tan^{-1}\frac{\theta}{2} \right\}.$$

$$\bar{k} = \frac{2a}{3} \left\{ \frac{\frac{1}{5}\tan^{-5}\frac{\theta}{2} - \tan^{-1}\frac{\theta}{2}}{\frac{1}{3}\tan^{-3}\frac{\theta}{2} + \tan^{-1}\frac{\theta}{2}} \right\} \quad \bar{k} = \frac{\frac{1}{5}\left(\cot^5\frac{\theta}{2} - \cot^5\frac{\theta'}{2}\right) - \left(\cot\frac{\theta}{2} - \cot\frac{\theta'}{2}\right)}{\frac{1}{3}\left(\cot^3\frac{\theta}{2} - \cot^3\frac{\theta'}{2}\right) + \left(\cot\frac{\theta}{2} - \cot\frac{\theta'}{2}\right)}$$

$$\bar{k} = \frac{2a}{3} \left\{ \frac{\tan^{-5}\frac{\theta}{2} - \tan^{-5}\frac{\theta'}{2} - \frac{1}{5}\left\{ \tan^{-1}\frac{\theta}{2} - \tan^{-1}\frac{\theta'}{2} \right\}}{\tan^{-3}\frac{\theta}{2} - \tan^{-3}\frac{\theta'}{2} + \right.$$

$$t = -\frac{a^2(1-e^2)}{h}\left\{\frac{e\sin\theta}{(1-e^2)(1-e\cos\theta)} + \frac{2}{(1-e^2)^{\frac{3}{2}}}\tan^{-1}\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right\}$$

$$t = -\frac{a^2}{h}\left\{\frac{e\sin\theta}{1-e\cos\theta} + \frac{2}{(1-e^2)^{\frac{1}{2}}}\tan^{-1}\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right\}.$$

Since

Compare the value of θ found by $\sin^{-1}$ and $\cos^{-1}$

mechane. $\sin^{-1}\sqrt{\frac{u}{\mu'}}\,x = \cos^{-1}\frac{y}{a}$

$$\text{also } \theta = \sin^{-1}\sqrt{\frac{u}{\mu'}}\cdot x \quad \therefore \sin\theta = \sqrt{\frac{u}{\mu'}}\cdot x$$

$$\text{also } \cos\theta = \frac{y}{a}$$

$$\text{also } \sin\theta = 1-\cos\theta = \frac{a-y}{a} = \sqrt{\frac{u}{\mu'}}\,x$$

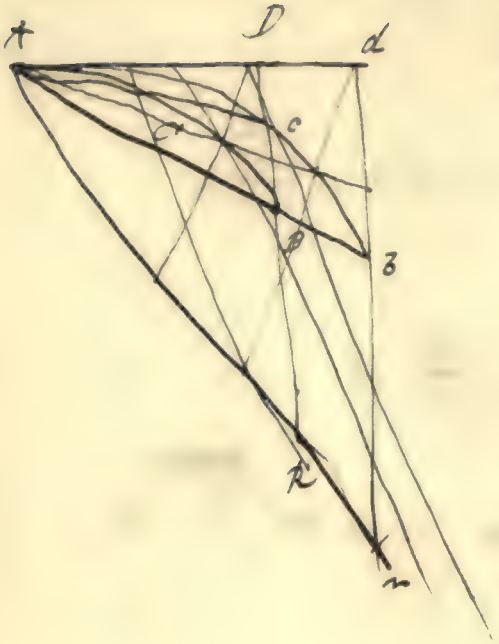
$$\therefore \frac{y}{a} = 1-\sqrt{\frac{u}{\mu'}}\,x \qquad y = a - a\sqrt{\frac{u}{\mu'}}\,x.$$

Cent. $v = \sqrt{FR}$. $P = 2\pi\sqrt{\frac{R}{F}}$

Cent. $v = \frac{\mu}{SP.RV}$. $P =$

$$T = \frac{\mu}{SP^2PV^3}.$$

Cent. diff. acc. $T \propto \frac{1}{PM^3}$


A
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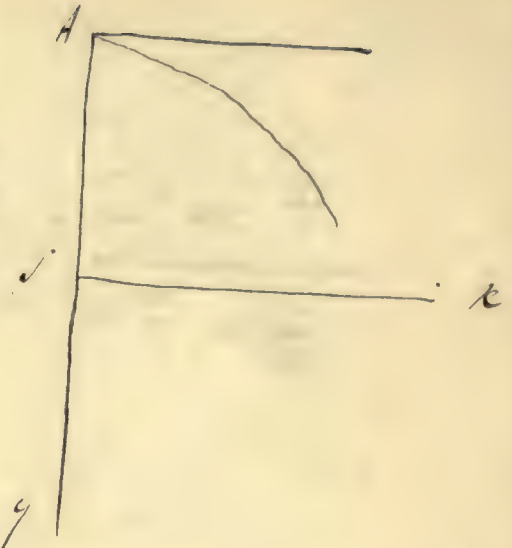
Let a heavy particle of mass m be projected from the origin O in the direction Ox with velocity V in a medium whose resistance per unit mass is μ times the velocity. Let Oy be the vertical through O in the plane of motion, and let g be the acceleration due to gravity.



Then the equations of motion are

$$\ddot{x} = -\mu\dot{x}, \qquad \ddot{y} = -g - \mu\dot{y}.$$

Integrating, with $\dot{x} = V$ and $\dot{y} = 0$ when $t = 0$,

$$\dot{x} = Ve^{-\mu t}, \qquad \dot{y} = -\frac{g}{\mu} + \left(\frac{g}{\mu} + V\sin\alpha\right)e^{-\mu t},$$

where α is the angle of projection. Integrating again, with $x = 0$ and $y = 0$ when $t = 0$,

$$x = \frac{V}{\mu}\left(1 - e^{-\mu t}\right),$$

$$y = -\frac{g}{\mu}t + \frac{1}{\mu}\left(\frac{g}{\mu} + V\sin\alpha\right)\left(1 - e^{-\mu t}\right).$$

Eliminating t between these two equations, we obtain the equation to the path:

$$y = \frac{g}{\mu^2}\log\left(1 - \frac{\mu x}{V\cos\alpha}\right) + \left(\frac{g}{\mu^2} + \frac{V\sin\alpha}{\mu}\right)\log\left(1 - \frac{\mu x}{V\cos\alpha}\right) + \frac{\mu x}{V\cos\alpha}\left(\frac{g}{\mu} + V\sin\alpha\right).$$

Let V_1 be the velocity with which the particle would strike the horizontal plane through O if it fell from rest through a height equal to the height due to the velocity V in vacuo. Then $V_1 = V\sin\alpha$, and the equation to the path becomes

$$y = \frac{g}{\mu^2}\log\left(1 - \frac{\mu x}{V\cos\alpha}\right) + \left(\frac{g}{\mu^2} + \frac{V_1}{\mu}\right)\log\left(1 - \frac{\mu x}{V\cos\alpha}\right) + \frac{\mu x}{V\cos\alpha}\left(\frac{g}{\mu} + V_1\right).$$

Let V_2 be the velocity with which the particle would strike the horizontal plane through O if it fell from rest through a height equal to the height due to the velocity V in the medium. Then $V_2 = V_1 - \frac{g}{\mu}$, and the equation to the path becomes

$$y = \frac{g}{\mu^2}\log\left(1 - \frac{\mu x}{V\cos\alpha}\right) + \left(\frac{V_2}{\mu} + \frac{V_1}{\mu}\right)\log\left(1 - \frac{\mu x}{V\cos\alpha}\right) + \frac{\mu x}{V\cos\alpha}\left(V_2 + V_1\right).$$

Let R be the range on the horizontal plane through O. Then $y = 0$ when $x = R$, and

$$0 = \frac{g}{\mu^2}\log\left(1 - \frac{\mu R}{V\cos\alpha}\right) + \left(\frac{V_2}{\mu} + \frac{V_1}{\mu}\right)\log\left(1 - \frac{\mu R}{V\cos\alpha}\right) + \frac{\mu R}{V\cos\alpha}\left(V_2 + V_1\right).$$

Let T be the time of flight. Then $y = 0$ when $t = T$, and

$$0 = -\frac{g}{\mu}T + \frac{1}{\mu}\left(\frac{g}{\mu} + V\sin\alpha\right)\left(1 - e^{-\mu T}\right).$$

Let V_3 be the velocity with which the particle would strike the horizontal plane through O if it fell from rest through a height equal to the height due to the velocity V in the medium. Then $V_3 = V_2 - \frac{g}{\mu}$, and the equation to the path becomes

$$y = \frac{g}{\mu^2}\log\left(1 - \frac{\mu x}{V\cos\alpha}\right) + \left(\frac{V_3}{\mu} + \frac{V_2}{\mu}\right)\log\left(1 - \frac{\mu x}{V\cos\alpha}\right) + \frac{\mu x}{V\cos\alpha}\left(V_3 + V_2\right).$$

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Let T be the time of flight. Then $y = 0$ when $t = T$, and

$$0 = -\frac{g}{\mu}T + \frac{1}{\mu}\left(\frac{g}{\mu} + V\sin\alpha\right)\left(1 - e^{-\mu T}\right).$$

$$\zeta = \frac{-c}{3\theta} \cdot \frac{r^3 \sin^4 \theta}{r^2}$$

$$\int_0 r^3 d\theta = \int_0 \frac{2m^3 \sin^3 \frac{\theta}{2} \cos \frac{\theta}{2}}{\sin^6 \frac{\theta}{2}} = \frac{2m^3 \cos \frac{\theta}{2}}{\sin^5 \frac{\theta}{2}} = \frac{2m^3 \cdot \sec^2 \frac{\theta}{2}}{\tan^3 \frac{\theta}{2} \cdot \sin^2 \frac{\theta}{2}}$$

$$= \int_0 \frac{2m^3 \cdot \sec^2 \frac{\theta}{2} \left(1 + \tan^2 \frac{\theta}{2}\right)}{\tan^5 \frac{\theta}{2}}$$

$$= 2m^3 \left\{ -\frac{1}{4} \tan^{-4} \frac{\theta}{2} - \frac{1}{2} \tan^{-2} \frac{\theta}{2} \right\}$$

$$a^2 - \frac{1}{3} \tan^{-3} \frac{\theta}{2} - \tan^{-1} \frac{\theta}{2}$$

$$= 2a \left\{ \frac{\frac{1}{4} \left\{ \cot^4 \frac{\theta}{2} - \cot^4 \frac{\theta'}{2} \right\} + \frac{1}{2} \left(\cot^2 \frac{\theta}{2} - \cot^2 \frac{\theta'}{2} \right)}{\frac{1}{3} \left(\cot^3 \frac{\theta}{2} - \cot^3 \frac{\theta'}{2} \right) + \left(\cot \frac{\theta}{2} - \cot \frac{\theta'}{2} \right)} \right.$$

$$V = \frac{2a}{3} \left\{ \frac{\frac{1}{2} \left(\cot^4 \frac{\theta}{2} - \cot^4 \frac{\theta'}{2} \right) + \left(\cot^2 \frac{\theta}{2} - \cot^2 \frac{\theta'}{2} \right)}{\frac{1}{3} \left(\cot^3 \frac{\theta}{2} - \cot^3 \frac{\theta'}{2} \right) + \left(\cot \frac{\theta}{2} - \cot \frac{\theta'}{2} \right)} \right. = GN.$$

Let $AV = a$. $\therefore AN = a + b$ and $NG = c$.

$(a+b) \cos \theta$ $P =$ $\therefore (a+b) d\theta =$ depth of cut of ζ.

below the surface.

$\therefore$ press = pressure at the surface $\times a + b \cdot d\theta$

Total pressure $= \int_0^{\theta'} r^2$ [illegible] the limit $= \frac{1}{2} \left\{ \frac{1}{3} \left(\cot^3 \frac{\theta}{2} - \cot^3 \frac{\theta'}{2} \right) + \left(\cot \frac{\theta}{2} - \cot \frac{\theta'}{2} \right) \right.$

$\therefore P = \frac{pf}{3} \left\{ \frac{1}{5} \left(\cot^5 \frac{\theta}{2} - \cot^5 \frac{\theta'}{2} \right) - \left(\cot \frac{\theta}{2} - \cot \frac{\theta'}{2} \right) \right\}$.

$$1 - \frac{V^2 R}{\mu}$$

$$1 - \frac{2V^2R}{\mu} + \frac{V^4R^2}{\mu^2} \quad \alpha^2\beta$$

$$\frac{V^4R^2}{\mu^2} + \frac{2V^2R}{\mu} + 1 \alpha^2\beta + C\alpha^2\beta = 0$$

$$= C\alpha^2\beta + \left(1 - \frac{V^2R}{\mu}\right)^2 \alpha^2\beta = 0$$

$$\beta = 0 \qquad 1 = \frac{V^2R}{\mu} \qquad V^2 = \frac{\mu}{R}$$

$$u = V^2 R.$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$$

$$\frac{du}{d\theta} d^2u + u\,du = \frac{\mu}{h^2} du$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2\mu}{h^2} u + C$$

$$\text{when } r = R. \text{ mechanic}$$

$$\alpha = \theta = \pm\beta \quad \therefore u\,du = -\pm\beta \quad du = -r\pm\beta$$

$$\left(\frac{d\theta}{du}\right)^2 = \frac{1}{R^2 - \alpha^2\beta}$$

$$\frac{C\alpha^2\beta}{R^2} + \frac{1}{R^2} - \frac{2\mu}{h^2 R} = C$$

$$\frac{1}{R^2\alpha^2\beta} - \frac{2\mu}{h^2 R} = C \qquad h^2 = R^2 V^2 \alpha^2\beta$$

$$\frac{1}{R^2\alpha^2\beta} = \frac{V^2}{h^2} = \frac{V^2 R}{h^2}$$

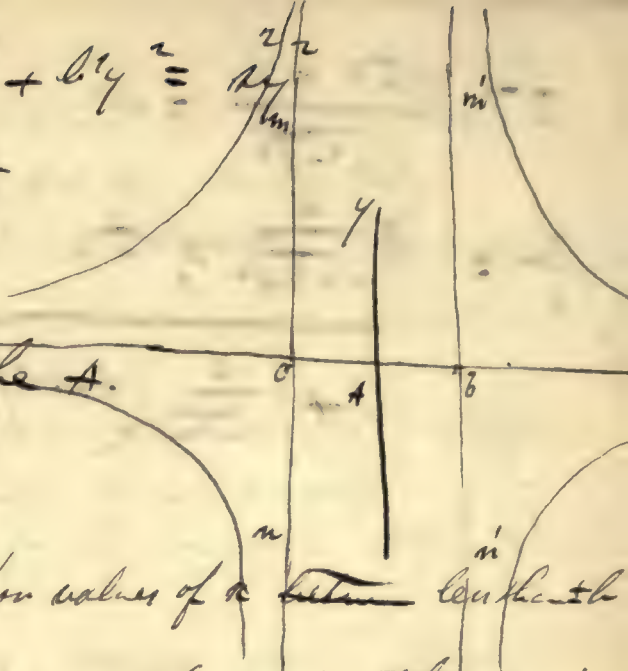
$$\therefore \frac{V^2R - 2\mu}{h^2 R^2} = C$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 = \frac{V^2R - 2\mu}{h^2 R} + \frac{\mu^2}{h^4} - \left(\frac{\mu}{h^2} - u\right)^2$$

$$\cos\theta + C' = \frac{\frac{u}{h^2} - u}{\sqrt{\frac{V^2R - 2\mu}{h^4 R} + \left(\frac{\mu}{h^2}\right)^2}}$$

$$\frac{1}{r} = \frac{u}{h^2} - \sqrt{\frac{V^2R - 2\mu}{h^2 R} h^2 + \left(\frac{\mu}{h^2}\right)^2}.$$

Lacette Curve whose $= {}^{n}$ is $a^2x^2 + b^2y^2 = x^2y^2$



since the $= {}^{n}$ contains only = powers of x and y the curve is symmetrically situated with respect to both axes. also since $x=0$ gives $y=0$ the origin is a point in the curve. letting be A. Ay. & the axes of y or x.

solving the $= {}^{n}$ with respect to y we have.

$$y = \frac{ax}{\pm\sqrt{x^2-b^2}}$$ which shows that for values of x between $\pm b$

y is impossible on the curve leaves the plane of the axis at the origin and does not again return to it until $x = \pm b$. in which case the value of y is ∞. wherefore for these values of x there are two asymptotes cutting the axis of x. at right $\angle$s. letting be m C n + m' C n' $\angle$s.

if $x = \pm\frac{5}{4}b$ $y = \pm\frac{5a}{3}$ if $x = \pm\frac{3}{2}b$ $y = \pm\frac{3a}{\sqrt{5}}$ $\angle$ —

if $x = \infty$ $y = 0$.

$$\frac{dy}{dx} = -\frac{ab^2}{(x^2-b^2)^{\frac{3}{2}}} = \infty \text{ if } x = \pm b.$$ ∴ the direction of the curve when it first returns to the plane. of the asymptote is $\perp$ to the axis of x.

$$\frac{dy}{dx} = -\frac{ab^3}{(x^2-b^2)^{\frac{3}{2}}} = 0 \text{ if } x = \infty.$$

wherefore the curve tends to become more nearly parallel to the axis of x as ab or inf did along the axis of x it meets it. in a cusp. the axis is a cusp. both in the +ve and neg direct. the curve consists of 4 branches as above.

$$\tan^{-1}\frac{a\tan x + b}{\sqrt{a^2-b^2}} = u$$

$$\frac{du}{dx} = \frac{\frac{a}{\sqrt{a^2-b^2}}\sec^2 x}{1+\left\{\frac{a}{\sqrt{a^2-b^2}}(\tan x+b)\right\}^2} = \frac{\frac{a}{\sqrt{a^2-b^2}}\sec^2 x}{\frac{a^2-b^2+a^2\{\tan^2 x+2b\tan x+b^2\}}{a^2-b^2}}$$

$$= \frac{a\sqrt{a^2-b^2}\cdot\sec^2 x}{a^2-b^2+a^2\{\tan^2 x+2b\tan x+b^2\}}$$

$$x^3+y^3-3axy = u = 0$$

$$\frac{du}{dx} = 3x^2-3ay$$

$$\frac{du}{dy} = 3y^2-3ax$$

$$\frac{dy}{dx} = -\frac{x^2-ay}{y^2-ax} = 0$$

$$\frac{du}{dx} = 0 \quad \therefore x^2-ay = 0 \quad \therefore x^2 = ay.$$

$$\frac{du'}{dx} = \frac{(y^2-ax)a+(x^2-ay)2y}{(y^2-ax)^2} = \frac{ay^2-a^2x+2xy^2-2ay^2}{}$$

$$\frac{du'}{dy} = \left\{\frac{(y^2-ax)2x+(x^2-ay)a}{}\right. \quad 2xy^2-2ax^2+ay^2-a^2y$$

$$\frac{dy}{dx} = \frac{2xy^2-ax^2-ay^2}{2xy^2-a^2x-ay^2}$$

$$\therefore \left(\frac{dy}{dx}\right)_1 = \frac{2\sqrt{ay}\,y^2-2a^2y}{2ay^2-ax^2-ay^2} = 2\left\{\frac{\sqrt{ay}\,y^2-a^2y}{ay^2-a^2\sqrt{y}}\right. = \frac{2(y^{\frac{3}{2}}-a^{\frac{3}{2}}y^{\frac{1}{2}})}{a^{\frac{1}{2}}y^{\frac{3}{2}}-a^{\frac{3}{2}}}$$

$\because x^2 = ay$

Matthew was a Galilean toll collector uncertain when he died some
assert by martyrdom at Kadabban Thus that he died in Persia or
in Parthia He wrote his Gospel about A.D 60 for the Jews
supposed to have been written originally in Hebrew - Irenaeus
first quotes this Gospel by name & Erasmus first questioned
its Hebrew origin & contended for the Greek
The names of Ahaziah Joash & Amaziah are probably omitted on
a/c of their wickedness as that of Simeon for his cruelty in the
blessing of Moses Joab in that of David for his murders & the Dan
in Revelations for their idolatry
πρωτότοκος is first born πρωτοτόκος bearing for 1st time
Bethlehem was a small town 6 miles south of Jerusalem in
the tribe of Judah Also called Ephrata Mic 5. 2 There was
another Bethlehem in Galilee where this is called B. in Judea
V 13 Egypt was near Bethlehem than Syria & Herods kingdom was
properly a part of Syria he probably had some influence with the
Governor - ~~Egt~~ Egypt (anciently called Mizraim) had now been
27 years under the Roman yoke
18 Was really fulfilled when the Jews were assembled at Ramah
to be led captive to Babylon Jer 40. 1 Ramah was the birth place
of Samuel in the tribe of Benjamin and in the confines of Bethlehem
was probably included in Herods massacre hence Rachel the mother
of Benjamin is said to weep for her offspring
22 After Herods death his kingdom was divided among his 3 sons
Archelaus Ethnarch of Judea Samaria & Idumea Antipas
Tetrarch of Galilee & Perea & Philip of Iturea & Trachonitis &
After reigning 10 years Archelaus was banished for his injustice to Rome.
Saul when he died A.D. 6 & his dominions became a Roman province
under a procurator subject to the Presi of Syria Coponius was the
1st procurator & Quirinius (Cyrenius in Luke) was then presi of Syria
4. Antipas had quarrelled with Archelaus & hence Joseph was
to safer in Galilee than in Judea
23 Nazareth was a small village in Lower Galilee

Ch 3. 3. This prophecy primarily related to the return of the Jews after the
edict of Cyrus B.C. 536
5 The Jordan rises in a.P. Phiala thence it passes under ground
& emerges again near Paneas then flowing southward thro' a. Phe
& the Lake of Galilee it at length loses itself in the Dead Sea Its
Course is about 100 miles
17 The word in Gr: is often used in the sense of Loves & thus
includes past present & future time ~~was~~ &c: This passage
is rendered "in whom I ever have been, am, and ever will be
well pleased" εὐδόκησα Hellenistically means to be pleased with or
approve of any one.
Ch: 4 5 πτερ... according to Heid was the roof of the great Eastern
porch the height of wh: was 385 feet
8 The kingdoms here mentioned were probably of Philip Antipas &c
Nero seems the most likely as by the Jews in Judea
12 John was imprisoned about a year after C. baptism &: about
18 months after the commencement of his own ministry - Betw: this event
& preceding is an interval of nearly a year during wh: Jesus went
into Galilee & visited Jerusalem - This Galilee is not the Gal: of
Herod but the coast of L. Tiberias in Philips dominions
13 Naz: was in the tribe of Zabulon & Capt: lay on the confines of
Zebulon & Nepthali & on the coast of the Sea of Galilee
14 Is: lived ~~B.C.~~ B.C. 760
18 Peter preached in the north of Asia Minor & became Bishop of Rome
& was crucified there by Nero about A.D. 69 Andrew preached in Scythia
& was crucified by Ageas proconsul of Achaia
24 Cœle Syria north of Upper Galilee
25 Decapolis was a country of Palestine in the half tribe of
Manasseh East of Jordan so called from its cont: 10 cities &
among the Greeks Doris was named Pentapolis
περαν του Jorin Perœa so called because it lay by
Jordan with eyes to Jerusalem it consisted of the tribes of
Reuben & Gad

Let x = hours run of till midnight. $\therefore 12-x$ = hours from noon

$\frac{3x}{2} - 12 = 4 - \frac{12-x}{2}$ $\quad \therefore 3x - 24 = 8 - 12 + x$

$2x = 20$

$12 - x = 12 - 10 = 2$ the hours from noon $\quad x = 10$

$x-3y, x-2y, x-y, x, x+y, x+2y, x+3y$ = the numbers.

Then $7x = 28 \therefore x = 4$

1. 2. 3. 4. 5. 6. 7.

Let $x+y$ = the number.

$x^2 + y = y^2 + x \therefore x^2 - y^2 = x - y$ or $x + y = 1$ = the number.

$(x-3y)^3 = x^3 - 9x^2y + 27xy^2 - 27y^3$

$(x-2y)^3 = x^3 - 6x^2y + 12xy^2 - 8y^3$

$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$x^3 = x^3$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+2y)^3 = x^3 + 6x^2y + 12xy^2 + 8y^3$

$(x+3y)^3 = x^3 + 9x^2y + 27xy^2 + 27y^3$

$\therefore$ the progression is 1. 2. 3. 4. 5. 6. 7

$7x^3 + 84xy^2 = 784$

$x^3 + 12xy^2 = 112$

$48y^2 = 112 - 64 = 48$

$y^2 = 1 \therefore y = 1$

The sum = 20 L + 2 shillings

4 times double = 40 L + 2Q shillings = 20 Q + 2 shillings

$39Q = 18Q \therefore 13L = 6Q$

Let $L = 6$ then $Q = 13$ sum $6 \cdot 13$ or any multiple of it

Let $L(x+y)$ be the binomial its coefft one $(1+1)$

i.th n^{th} power $= (x+y)^n$. the nth powers of its coefft $= (1+1)^n = 2^n$

p^{th} coefft $= \frac{n(n-1)(n-2)\ldots(n-p+2)}{1.2.\ldots(p-1)} \cdot x \cdot y^{n-p}$.

4th Term of $(a^{\frac{2}{3}} - b^{\frac{3}{2}})^{\frac{4}{5}} = \frac{\frac{4}{5}(-\frac{1}{5})(-\frac{6}{5})}{1.2.3.} (a^{\frac{2}{3}})^{\frac{4}{5}} \cdot -4 (b^{\frac{3}{2}})^4 = \frac{4}{125} \frac{b^6}{a^{\frac{22}{15}}}$

$= \frac{4.1.6}{0.5^3} \left\{ -\frac{22}{15} \right\} a \cdot b^6$

Trig Problems.

St. Johns. Dec. 1833.

1) Find the number of degrees, minutes and seconds in $\frac{180^\circ}{\sqrt{3}}$ & shew that $\sec 2A = \frac{\sec^2 A}{1-\tan^2 A}$

2. Find the value of $a\cos 2\theta + b\sin 2\theta$ when $\tan\theta = \frac{b}{a}$, and determine x from the equation $\text{vers}^{-1}(1+x) - \text{vers}^{-1}(1-x) = \tan^{-1} 2\sqrt{1-x^2}$

3. Transform the equation $\sqrt{2\sec A} = \frac{1}{5}\sqrt[3]{\frac{\cos^2 B}{\sin C}}$ into the equation between the tabular logs of the quantities.

4. In Euclid B III. prop X shew that the areas of the circles are as $5+\sqrt{5} : 2$.

5. Prove in the case of the sine and sec that the log sine varies generally as the angle.

6. The distance between the centers of two wheels = a and the sum of their radii = c shew that the length of a string which crosses between the wheels and goes round them $= 2\{\sqrt{a^2-c^2} + c\cos^{-1}\left(-\frac{c}{a}\right)$

7. AB and AC are the chords of arcs of 60° and 90° in a circle whose center is O. Shew, if AB and OC be produced to meet in D that the arc whose chord = DC has its cosine & chord equal.

8. If the circle inscribed in triangle ABC touch the sides AB, BC, CA in P, Q, R, and $s = \frac{1}{2}(a+b+c)$ shew $AP^2 + BQ^2 + CR^2 = (s-a)^2 + (s-b)^2 + (s-c)^2$

9. If $s - x = nd - (a+x)$ expand x in a series of the form

$$Ad x + Bd x 2A + Cd x 3A + \&c$$

and prove that if n be any odd number

$$\tan\frac{\pi}{4n} \times \tan\frac{5\pi}{4n} \times \tan\frac{9\pi}{4n} \cdots \tan\frac{(4n-3)\pi}{4n} = 1$$

10. The circumference of the inner of two concentric circles of which R and r are the radii is divided into n equal parts, shew that the sum of the squares of the lines drawn from the point of division to any point in the circumference of the ~~circle~~ outer circle $= n(R^2+r^2)$

St Matthew

5. Chap VIII. v4. δὲ αὐτοῖς ἐντείλατο what was the reason of this injunction? v11. ἀπὸ ἀνατολῶν καὶ δυσμῶν does the δύσις refer? v16. ὀψίας δὲ γενομένης Supply the ellipsis. Make how many evenings the Jews reckoned, & their division of the night in the time of our Savior? Explain v22. & point out what distinguishes the miracle related at the end of the chapter from all the other miracles wrought by Christ.

6. Chap IX. v10. τελωνῶν καὶ ἁμαρτωλῶν. How was the tribute money collected by the Romans in Judea? What made the Jews rank the publicans with sinners? Give the literal meaning of κλίνη & explain v6 & 7. v20. What part of the Jewish garments is meant by κράσπεδον? v23 In what respect did the Jews in general & their dead differ from the other nations of the earth? v34 How that the opposition of the Jews to Xt. confirms the truth of the Gospel miracles. v36 & 38 Paraphrase Do you recollect any passage where Cicero uses the word expandere in the same sense?

10. Chap X. v3. Ἰάκωβος ὁ τοῦ Ἀλφαίου — Ἰάκωβος ὁ τοῦ Ἀλφαίου Which of these was the author of the epistle of St James? Reconcile v10 with Mark ch VI. v8. v16 What is the derivation of πανοῦργος? v21. What argument does this verse afford for the truth of Xty. Explain v34 & 35 What custom is alluded to in v36 v41 How do you understand the phrase εἰς ὄνομα προφήτου

11. Give the primary significations of μεθοδεία, διδασκαλία, ἐναγγελίζω, βασιλεία, βαπτίζω, & point out the difference of signification between θέλημα & ἐντολή — ἐπαναπαύομαι and ἐφίστημι; ἐξουσία & ἐξουσιάζω, ἀναίρω & διαβολος-

12. Is the expression βασιλεία τῶν οὐρανῶν peculiar to St Matthew?

13 Translate into Latin v27 Chap 10 & six following verses.

14 Draw a map of Palestine, so as to represent the relative boundaries of Samaria, Lower & Upper Galilee, the Idumæan, of Judea, and Decapolis, & mark on it the towns of Jerusalem, Nazareth, Rama & Bethlehem?

St. Matthew.

I. What do you mean by the authenticity of the book? and how would you proceed to demonstrate the authenticity of the books of the New Testament?

II. What was the language spoken in Judea in our Saviour's time? Give instances of the use of it? Mention any Hebraisms which occur in the first 10 Chapters?

III. Mention briefly instances in which the Gospel narrative receive illustration from the peculiar customs of the East. as to marriages, treatment of the dead made of building houses. headmen of ships.

IV. Define exactly the boundaries of the Jurisdiction of the Herods dom? What promise did God make promise to Joseph? Why is Herod sometimes called K. of the Jews? Herodotus.

V. What are the different names of the Sea of Galilee. What the country in the east?

VI. Reconcile these apparent contradictions

Mark XV. 22 with Mark XVII. 24

Mark XV. 32 with Mark VIII. 10

VII. Quote the passages in the first 10 Chap. of St. Matthew that have been the chief cause of your Saviour.

VIII. Give Doddridge reflection on Matth. XII. 43. also the observation of Herod on hearing the fame of Jesus. — on Matth IX. 16.

IX. Define a Sacrament. and then how the Christian Sacrament answers your definition.

X. Shew that our Lord contemplated his passion from first to last during his ministry.

XI. Shew that the germ of all the leading doctrines of Christianity is to be found in the Gospels.

XII. Translate the words βαπτισθηναι. επιουσιον, ορος, κλεος, δοκος, αγελαι, κελευω, πεσσαλος, ῥιζα, γενεθλια, κλαυσον, ψαλλον, ιδεσθαι, χρηματιζω, δελτος, της πεδαρσεως, σκανδαλιζω, πεζευσωμεν.

XIII. What is the value of the didrachmon, and the στατηρ, for what purpose was the former collected. Chap XVII. 24.

XIV. Arrange in chronological order the Chapters of St. Matthew which relate (1) to the first year of our Lords ministry, (2) to the second year of the same.

XV. Translate into Greek Prose —

S. Matthew.

State the argument in favour of the authenticity of the books of the New Testament from the character of the language in which they are written. Specify and illustrate some of its most striking peculiarities.

Chap I. 1. Ἰησοῦ Χριστοῦ What is the signification of this title? How came it to be used as a proper name. Was it ever so used during our Saviour's life time.

Chap II. 15 Rosenmüller and others enumerate four significations of the phrase ἵνα πληρωθῇ in the New Testament give them. Shew from the contents of this gospel that it was originally designed for the use of the Jewish nation.

III. v. 10. γῆν Ἰούδαίαν What track of country did this embrace? State the original name and extent of the holy Land, with the political divisions made by Joshua?

V. Chap III. 9, 10. Translate and give a full explanation of these verses.

VI. v 17. What is the force of the article prefixed to ἀγαπήσεις. Why is the verb in the first aorist? Are there instances of this in the gospels? -

VII. Chap IV. 3 Explain the phrase υἱὸς τοῦ θεοῦ. Should a son of God be a more correct translation?

VIII. v 18. How may this account of the call of Simon and Andrew be reconciled with that which S. Luke gives?

XI. v 23 Give an account of the Jewish synagogues. What was the nature of the ecclesiastical power possessed by them? - What instances of this being enforced are mentioned in the writings of the New Testament?

X. Chap VIII. 3 Give an account of the principal regulations prescribed by the Mosaic law respecting this disease.

XI. 7 v 11 & 12. Explain these verses. Quote the principal prophecies of the Old Testament which refer to the calling of the Gentiles.

XII v 19. Prove the antiquity of the Jewish method of beginning the day observing what is meant by ὀψὲ σαββάτων? -

XIII. v 22. Explain this. What reason may be assigned for our Saviour's frequent use of metaphorical language?

XIV. Chap IX. 12. 13. Translate these accurately. State the proof of the divine origin of our religion from the purity of its precepts.

XV v 14. Did the Jewish law enjoin the observance of fasts.

XVI. 16. Illustrate the custom here referred to by reference to sacred and profane authors.

Conic Sections

(I) What do you mean by the rectangular co-ordinates of a point; what by the polar co-ordinates?

(II) Find the distance of two points from each other in terms of their co-ords I° when the system is rectangular II° when the system is oblique; also find the inclination of the line that joins them to the axis of x.

(III) Given $r = a$, $\theta = x$ the polar co-ordinates of P find the rect° co-ord^{ts} what is meant by equation to a curve, and the locus of an equation. Prove the following equation to a straight line

$$y = mx + c \;;\; y = -\frac{c}{a}x + c \;;\; y \sin\alpha + x\cos\alpha = p$$

and construct a straight line whose equation is $y = -mx - c$

Find the equation to a straight line which shall pass through two given points, whose co-ordinates are x', x''; y', y''; also to a straight line which shall pass through a given point and be parallel to a given straight line.

Given the equations to two given straight lines - determine the points of their intersection - and to find the angle between them

Find the equation to a straight line which passes through a given point and makes a given angle with a given straight line

Given the coordinates of a point, and the equation to a straight line to find the length of the perpendicular dropped from the point upon the line.

Find the equation to a straight line referred to oblique co-ordinates. Prove the polar equation to a straight line to be $r = p \sec(\theta - \alpha)$ also that $A\cos\theta + B\sin\theta + C = 0$ is the polar equation to a straight line.

Conic Sections

If a right cone be cut by a plane, the section will be an ellipse a parabola or an hyperbola.

In the parabola if a is axis PV be drawn through any point P and be intersected in V by any ordinate QVq drawn parallel to the tangent at P then will $4SP \times PV = QV^2$ SP being the focal distance.

In the ellipse the rectangle contained by the perpendiculars from the foci upon the tangent is invariable; and the rectangle contained by the focal distances of any point is equal to the square of the semi-diameter drawn parallel to the tangent at that point.

If from the point of intersection of two tangents a diameter be drawn to the ellipse it will bisect the straight line joining the points of contact.

In the hyperbola the difference of the squares of any two conjugate diameters is constant and the areas of the parallelograms formed by tangents at the extremities of any pair of conjugate diameters is constant.

The chord of curvature which passes through the focus of an ellipse $= \frac{2CD^2}{AC}$

PSp is drawn through the focus S of a parabola Pp whose vertex is A, and AP AP are joined: shew that $3 \tan PAp + 4 \operatorname{cosec} ASP = 0$

Explain the principle of prime and ultimate ratios: and shew that the ultimate ratios of the sine, arc and tangent of a circular arc are ratios of equality.

Explain what is meant by continuous and finite curvature in curves; and shew that in curves of finite curvature, the difference of the angle of contact is ultimately as the square of the contemporaneous arc.

Enunciate and prove Newton's 1st Lemma.

When a body moves in a curve acted on by forces tending to a fixed point, the areas which it describes by rays drawn to the centre of force are in a constant plane, and are proportional to the times.

When a body revolves in a curve about a fixed centre of force, find the general expression for the value of the force.

The periodic times in all ellipses round the centre is constant, if the intensity of the force is constant.

A body is projected from a given point with a given velocity in a circle at the same distance, find the magnitude and position of the axes of the orbit when the force varies as the distance. If when the body has arrived at the extremity of the greater axis the intensity of the force were altered in the ratio of $n:1$ find the change produced in the orbit.

A body revolves in an ellipse, find the force tending to the focus of the ellipse. Also shew that the periodic time in the ellipse varies in the sesquiplicate ratio of the axis major.

St. Matthew.

1) State briefly your opinion and the grounds of them respecting the period about which this gospel was composed, and whether the Greek is the original or a translation. Is it probable that there were two originals.

2) After the death of Herod, and banishment of Archelaus by whom were the Jews governed? were there governors appointed by the Roman Senate only the emperor? what degree of civil and religious liberty did the Jews possess under them.

3. Describe the official authority of Rabbi after the synagogue? who were the scribes mentioned in the New Testament? in what respect did they differ from the scribes mentioned in the Old Testament?

4. Chap I. what is the literal signification of Χριστος? To whom besides our Saviour is this term applied in the old Testament? which of the prophets more particularly apply it to our Saviour? Can you shew by passages in this gospel that the Jews expected Christ to appear as a king? translate v 19. and assign reasons for the way you render θεασθαι and ὀναινομαι

5. Chap II. Give a short account of the Herod here mentioned, and shew that the slaughter of the infants agrees with the character given of him by Josephus. v 11. To whom is the title of Χριστος applied in the gospel v. 10. Ἐκεινον. Χαιρω ὑπερ τῆν φωνῆν τοῦ Θεοῦ do you recollect a similar expression of any later author.

6. Chap III. v 1. what two offices did the Jews call wilderness? what part of Judea do you understand by Ἐρημος τῆς Ἰουδαίας? Shew that baptism was in use among the Jews before the coming of John. and point out the difference between the baptism of John, and that of Jesus Christ. Translate v 7 and state in what sense you understand γεννηματα ἐχιδνων translate v 11. & 12 and explain the phrases βαπτιζω ἐν ὑδατι εἰς μετανοιαν and βαπτιζω ἐν πνευματι ἁγιῳ καὶ πυρι.

7. Chap IV. State the reason which? Paul assigns for God's suffering Ch. to be tempted. and then for the account given of the third temptation. that it was not founded on a vision. translate v 10 Do you recollect any passage in a Latin author where audito is used in the same sense as ἀκουσῃ in v 24

Prove that if $2\cos\theta = x + \frac{1}{x}$ then will $2\cos n\theta = x^n + \frac{1}{x^n}$ and $\sqrt{-1}\sin n\theta = x^n - \frac{1}{x^n}$ whether (n) be integral or fractional

Hence deduce Demoivre's theorem

Prove that $\sin\theta = \theta - \frac{\theta^3}{6}$ very nearly when θ is small.

Apply this formula to find the numerical value of $\sin 1^\circ$

Expand $(\cos\theta)^m$ in a series of cosines of multiples of θ

Ex. $(\cos\theta)^4$

Given $\tan\phi = \frac{n\sin\theta}{1 - n\cos\theta}$ find ϕ in terms of θ, (n) being a small fraction

Explain the method of forming a table of the natural sines and cosines of all angles from 0° to 90°

Show that a small angle cannot be found from its logarithmic sine

How is this difficulty obviated

Plane Trigonometry

Trinity College

1. Define the sine cosine and tangent of an angle. Trace the changes in magnitude and algebraical sign of the sine and cosine and tangent through the four quadrants

2. If α be the arc of a circle subtending an angle A shew that $\alpha = \frac{\pi}{180^\circ} \cdot R.A$

Find the number of degrees in an arc equal to the radius of the circle.

3. Prove the formulæ

(1) $\tan A = -\cot\left(\frac{3\pi}{2} + \alpha\right) = \tan\{(2n-1)\pi + A$

(2) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(3) $\tan(45^\circ + B) = \tan(45^\circ - B) + 2\tan 2B$

(4) $\frac{\cot 3\theta - \cot\theta}{\cot 3\theta + \cot\theta} = -\frac{1}{2\cos 2\theta}$

4. In any triangle the sides are proportional to the sines of the angles opposite to them

Also, $\sin A = \frac{2}{bc}\sqrt{s.(s-a)(s-c)}$.

5. Given two sides and the included angle solve the triangle

In a triangle ABC given $AB = 345$ yards $AC = 174.07$ and the angle $A = 37^\circ.20'$. Required the other angles

$\log 51.907 = 1.7152259$, $\log \tan 71^\circ.20' = 10.491$

$\log 17.093 = 1.2328183$, $\log \tan 44^\circ.16' = 9.9880$

From the top of a mountain $2\frac{1}{3}$ miles high an angle is taken between the horizontal line and a line conceived to touch the earth and observed to be $88^\circ.2'$. Determine

Plane Trigonometry

St. Johns. June 1840

(I) Given the circular measure of an angle, shew how to find its measure in degrees, and conversely find the number of degrees in the angle whose circular measure is 2; and the circular measure of the angle which contains 28° 3′ Mention some advantages of the circular measure.

(II) Prove the following formulæ. $\operatorname{cosec}^2 A = 1 + \cot^2 A$; $\tan A = -\tan -A = \tan 180° + A$

$\sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B)$; $\sin 4A = 8\cos^3 A.\sin A - 4\sin A.\cos A$.

III. Investigate a general formula including all angles which satisfy the equation $\sec A = -2$

IV. Having given $\sin(-330°) = .5$ Hence deduce the value of $\sin(-164°)$ and account for $\sin A$ when determined from $\sin 2A$ having 4 values and no more.

V. Prove that $\dfrac{\cos A + \cos B}{\cos B - \cos A} = \dfrac{\cot \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$

VI. Shew how to find the value of $\sin 10''$; and how the sines of all angles from 0 to 90° may be computed at intervals of 10″ – How is the accuracy of the work at any particular step tested?

VII. Prove that in general the alteration in the cosecant for a given small alteration in the angle varies as the alteration in the angle. State the cases of exception.

Given L.cos 20°.35′.20″ = 9.9713351 / L.cos 20°.35′.10″ = 9.9713430. Find also L.cos 9.971338.

VIII. Given the hypothenuse of a right angled triangle = 1000 ft. and 1 side = 1 foot; give the best logarithmic formula for finding the opposite angle of the triangle

IX. Shew that in any triangle the sines of the angles are to one another as the sides opposite.

X. Given the three sides of a triangle, find the tangent of half an angle; and hence shew that no side of a triangle can be greater than the other two.

XI. Assuming Demoivre's Theorem, obtain $\tan\{(2n+1)\theta\}$ in terms of $\tan\theta$; finding the final terms in the numerator and denominator.

XII. Obtain $(\sin\theta)^{4m+3}$ in terms of series of multiples of θ; find the value of $(\cos\theta)^{\ldots}$

XIII. Shew how to solve the equation $x^3 - qx - r = 0$ by Trigonom. Ex $x^3 - 6x - 4 = 0$.

XIV. Define a logarithm; and shew that $\log_a \frac{m}{n} = \log_a m - \log_a n$; prove also that $\dfrac{\log_a^2}{\log_a^5} = \dfrac{\log_a^5}{\log_a^2}$; and find the log of 8 to the base $\frac{1}{2}$

XV Expand e^x in a series of ascending powers of x

Show that are whose radius = rad contains 57.°29577, and that its length is independent of r

What is meant by θ. Construct an angle whose cosec = d.

Express the cosine of an angle in terms of its sine, and hence all the trigonometrical ratios in terms of the sine and cosine.

Ex. 45°, 30°.

Prove the following $\sec^2 A = 1 + \tan^2 A$; $\sin A = \frac{\tan A}{\sqrt{1+\tan^2 A}}$

Trace the variations in the values and signs of the trigonometrical ratios as θ advances from zero to 2π.

Express the sine, cos and tan of 475° in terms of sine, cos and tan of an angle less than $\frac{\pi}{2}$

Give an instance of the utility of affixing contrary signs to lines drawn in opposite directions.

If one angle is supplementary to another their trigonometrical ratios are equal and of contrary signs except the sine and cosec and which are equal and of the same sign.

If two angles are equal but of contrary signs then all their trigonometrical ratios are equal and of contrary signs except the co-sine and ~~co~~-secand, which are equal and of the same sign.

Prove that $\sin n\theta = \sin\{n\pi + (-1)^n \theta\}$

$\cos n\theta = \cos\{(-1)^n \cos n\pi \pm A\}$ and find all the values of θ which satisfy the equation $\operatorname{cosec}\theta = d$.

Show how degrees may be converted into grades and the contrary. How many degrees and grades are contained in an angle whose circular measure = α

Ex. .7854

Differential & Integral Calculus -

St. John's Coll - May 1837

Prove that the differential coefficient of $f(x)$ with respect to x is the limit obtained from $\frac{f(x+h)-f(x)}{h}$ by making $h=0$; and hence deduce the diff coeffs of $\frac{x^2-1}{x}$ and $\cosh x$

Differentiate a fraction of which the numerator and denominator are functions of x; also a^x, $\sin x$, and $\tan^{-1} u$, u being any function of x

Differentiate the following expressions

$\sqrt{ax+x^2+\sqrt{a^4-x^4}}$, $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ $\log_e\{\cos x - \sqrt{-1}\sin x\}$

$c \sin x \sqrt{1-c^2\sin^2 x} + \sin^{-1}(c\sin x)$

Prove Maclaurin's theorem, and apply it to expand $\frac{1}{(1-x)^{\frac{m}{n}}}$

Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, y being a function of x determined by the eqn $x^4+2ax^2y-by^3=0$, and investigate the results.

Find the value of a fract. whose numerator & denom. both become $=0$ for a particular value of x. Shew what other expressions can be reduced to the same form; and find the values of $\frac{a^x-b^x}{x}$ and $x^m\left(\log_e\frac{1}{x}\right)^n$ when $x=0$ and of $\left(1+\frac{a}{x}\right)^x$ and $x\{\sqrt[x]{a}-1\}$ when $x=\infty$

Prove that $\tan PTX = \frac{dy}{dx}$, PT being a tangent to a curve at the point x, y — when does a tangent become an asymptote? Ex $y=\frac{x-3}{(x-1)(x-2)}$ find where the tang is parl. to the axis of x and also what $\angle$ the curve cuts the axis;

Shew how to pass from rectangular to polar co-ordinates with the same or a different origin — Ex $y^2=(1-e^2)(a^2-x^2)$ to the polar eq. with origin $x=ae$, $y=0$; absolute & rel. values — the radius vector — & the perpendicular on a tangent.

Find the expressions for the radius of curvature, and chd of curvature through the origin, at any point of a spiral. Ex Epicycloid. when the circles are equal.

Investigate the integrals of ax^{n-1}, $\frac{1}{x}$ and of $\frac{dx u}{u}$, u being a function of x; & integrate

$(9x^3+3x^2)^n(1+x^2)$; $\frac{2-4x}{\sqrt{x^2-x-2}}$; $\frac{x^2-1}{x-x^2+x^3}$; $\frac{1}{a^2+x^2}$;

Investigate the integrals of $\frac{dx u}{a^2\pm u^2}$, $\frac{dx u}{\sqrt{a^2\pm a^2}}$, $\frac{dx u}{u\sqrt{u^2\pm a^2}}$

and integrate $\frac{x^{2n-1}}{1-x^{2n}}$; $\frac{1}{\sqrt{(x+2a)(a-x)}}$; $\frac{1}{x\sqrt{a+bx^n}}$; $\frac{1}{(c+ex^2)\sqrt{a+bx^2}}$

State the four forms to the integration of which the integration of every rational fraction may be reduced; and integrate

$\frac{1}{x^3+x^2-2x}$; $\frac{1}{x^3+4x^2+4x}$; $\frac{1}{(x^2+3)(6^2+5)}$; $\frac{1}{(x^2+1)^2(x^2+4)^2}$; $\frac{x^3}{\sqrt{2ax-x^2}}$; $\tan mx$

Integrate $(a-x)^n$ from $x=0$ to $x=\frac{\pi}{2}$; and $\frac{\sin(mx)}{e^{ax}}$ from $x=0$ to $x=\infty$

also find the volume, and area of the surface of a paraboloid, cut off by a plane through its focus —

Theory of Equations

Dec. 1837.

I If a and b be taken substituted for x give results with different signs one root at least lies between them; and if a and every greater quantity give a positive result, a is greater than the greatest root.

Ex $6x^3 - 14x + 263 = 0$ has a root between 2.7 and 2.8

$x^4 + 11x^2 - 26x - 98 = 0$. 3 is greater than the greatest root.

II. To find the greatest and least values when $x^n + p_1 x^{n-1} + \&c + p_n$ is divided by $x - a$ Ex $x^4 - 2x^3 - 6x^2 + x - 11$ divided by $x - 4$

III. Transformation of eq^n into one whose roots are the reciprocals of its roots; and point out those cases in which the original and transformed eq^{ns} are identical.

IV. No eq^n complete or incomplete can have more positive roots than it has changes of signs; and no complete eq^n can have more negative roots than it has continuations of the same sign.

Ex $x^5 + x^4 + x^2 - 25x - 36 = 0$ find the limit to the number of positive and negative roots.

V. Every number which written for x makes $f(x)$ and all its derived functions positive, is a superior limit of the positive roots of $f(x) = 0$

Ex $x^5 - 10x^3 + 6x + 1 = 0$, find the integer next greater than the greatest root.

VI. The imaginary roots of $x^n - 1 = 0$ n being a ~~positive~~ prime no. are the several powers of a pair 1 to $n-1$, any one of the imaginary roots; to the eq^n $x^6 - 1 = 0$ which root produces by its powers the other roots; find three values of $\sqrt[3]{8}$.

VII. The eq^n $f(x) = 0$ has equal roots or not, according as $f(x)$ and $f'(x)$ have a common measure or not. What relation between m and n will make $m^2 x^2 = 2(n-x)^2(1-x^2)$ have equal roots?

VIII. Give Newton's method of approximating to the root of an eq^n [illegible] root of the eq^n $x^3 - 2x - 5 = 0$, to four places of decimals determine the conditions to be satisfied in order that the successive [illegible] may be certain?

XI. Give Cardan's solution of a Cubic, and shew in what cases it can be applied.

X. If $\frac{p}{q}, \frac{p'}{q'}$ be two consecutive fractions converging to any quantity x, prove that $x - \frac{p}{q} < \frac{1}{q^2}$ > $\frac{1}{2q^2}$

Ex. find the difference between $\sqrt{2}$ and the fourth convergent fraction approximating to $\sqrt{2}$.

Gnostics. doctrine. The doctrines of the world was divine. They denied the authority of the books of the Old Testament, because they were opposed to the account of the creation which the Jews cried to. They denied the deity of Christ, considering him inferior to the father — and they regarded his humanity upon the supposition that every thing corporeal and compound is in itself essentially and intrinsically evil. They asserted that Moses was only actuated by the malignant genius of this world.

Gnostics. morals — There were two sects — the one resembled the monks in ascetic manners — the other — to whom all the actions were alike in a moral point of view, and thus confound right with wrong, they gave a loose rein to all their passions and affected the innocence of living according to their own inclinations, as a following blindly their Lawmakers dictates. For the authority they referred to fictitious writings of Abraham. Jaeobites &c. affected boasting of having drawn these opinions from certain secret doctrines of Christ which were not exposed to vulgar eyes; others, that they had arrived at these ideas of condemnation by an innate force and vigour of mind.

Condemned by Christ — The Nicolaitanes, spoken of with abhorrence by Christ were a branch of the Gnostics, the licentiousness of their practice being openly confessed of — Corpocrates, a Gnostic leader of the second century, not only allowed his disciples full liberty to sin — but recommended them a vicious course of life as a matter both of obligation, and necessity, asserting that eternal salvation was only attainable by those who had first committed all sorts of crimes, and had daringly filled up the measure of their iniquity. That it was the will of God that all things should be possessed in common, but that women ought to be married and adulteries, to be only made use of — natural rights.

Thus did it happen to men who, forsaking the beaten track of scripture, endeavour to establish for themselves a particular sect. They may be the original founders of the schism — do not always, but do put into practice abominables as the followers adopt, by their opening the door of conduct — and perpetuation they must surely take a grievous responsibility on themselves —

-cydides . Born 77 Olympiad. B.C. 470. His history comprehends
-fifth Year of the Peloponnesian war.

Cubic. Theorem. Each alternate term in a Cubic Series
-ginning the 2nd after the series $a+bx+cx^2+dx^3+\&c$
Let the terms of the series be called A. B. C. D &c
and let $C = fB + gx^2A$, $D = fC + gx^2B$.
where $f+g$ is the Scale of relation

Then $A = A$
$B = B$
$C = fB + gx^2A$
$D = fC + gx^2B$ &c
and $S = A + Bx + fxA - A + gx^2xS$ &c
$S = \frac{A + Bx + fxA}{1 - fx - gx^2}$

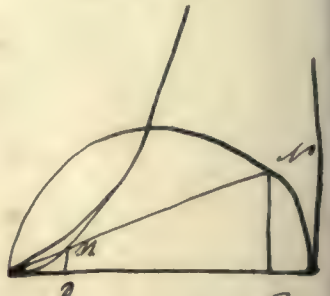
Ex. $1 + 2x + 3x^2 +$ &c
Here $f = 2$. $g = -1$
$\therefore S = \frac{1 + 2x - 2x}{1 - 2x + x^2} = \frac{1}{(1-x)^2}$

Ex 2. $1 +$ [illegible]
Here $3f + g = 5$ and $5f + 3g = 7$
$\therefore f = 2$. $g = -1$
$S = \frac{1 + 3x - 2x}{1 - 2x + x^2} = \frac{+x}{(1-x)^2}$

The equation $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ may be considered as the equation to any conic section: for the eq. to the ellipse b^2 is positive, to a parabola b^2 is infinite, and to an hyperbola b^2 is negative

To find the eq. to the Cissoid

Let AB be the diameter of the circle ARB. from the point R and B taken at equal distances from A and B draw RN, BM right angles to AB and join AR meeting BM in M. the point M will trace out a curve called the Cissoid of circles. From the nature of the circle $AR \times RB = RN^2$ and by the construction $AR \times RB = BM \times AP$. also from the similar triangles APM, ARN, $AP : PM :: AR$ or $AP : PM :: BP : RN = \frac{BP \times PM}{AP}$



also $RN^2 = \frac{BP^2 \times PM^2}{AP^2} = BP \times AP$

Therefore $BP \times PM^2 = AP^3$ Let $AB = b$, $PM = y$, $AP = x$

Then $(b - x)y^2 = x^3$ is the eq. required

Trigonometrical Problems

I. Shew how to find the circular measure of an angle from the number of degrees, minutes, & seconds which it contains; and having given the angle which the earth's radius subtends at the sun, find the circular measure of this angle & thence the distance of the sun from the earth

II. Investigate the following expressions for $\tan\frac{\theta}{2}$

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} \pm \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \quad \text{and} \quad -\frac{1 \pm \sqrt{1+\tan^2\theta}}{\tan\theta}$$

and explain why the latter one has only two values.

III. Explain the notation $\sin^{-1}$, $\tan^{-1}$ &c for the inverse circular & prove that $\tan^{-1}\frac{x\cos\varphi}{1-x\sin\varphi} - \tan^{-1}\frac{1-\sin\varphi}{\cos\varphi} = \varphi$

IV. Eliminate θ between the eqns.

$$\tan^3\left\{\frac{\sin^{-1}(3\sin\theta)+\alpha}{4}\right\} = \tan\left\{\frac{\sin^{-1}(3\sin\theta)-3\alpha}{4}\right\}$$

$$\cos(\varphi-\theta+\alpha)\cdot\cos(\theta-\alpha) = \cos(\varphi-\theta-\alpha)\cdot\cos(\theta+\alpha) = m$$

V. If the tangents of the angles of a plane triangle be in geometrical progression whose ratio is n; shew that $\sin 2C = n\sin 2A$.

VI. Reduce the negative logarithm $-\{1.8753145\}$ to another of which the characteristic only shall be negative: and then find the corresponding number having given $\log 1.3325 = 1.246672$, and $\log 1.3326 =$

VII. If O be a point within a triangle ABC from which the sides subtend equal angles, find S and S' the sum and diff of the squares of the distances from the angular points to points in lines of the sides; and shew that when the triangle is right $\angle ABC$ that $S = c^2 - \frac{ab}{\sqrt{3}}$, and $S' = \sqrt{c^4+a^2b^2}\sqrt{3}$

VIII. If $\text{cotangent}\,\theta - \cot\alpha = 2\sec 2\varphi$ and $\cot\theta' - 2\sin\alpha' = \tan^2\varphi$ } such constant values may be given to α & α' as shall make $\tan\theta - \tan\theta'$ numerical

IX. $\cos n\theta = \cos\alpha$. Shew that all [illegible] values of $\cos\theta$ which satisfy this eqn will be included in the formula $\cos\frac{2\pi\lambda+\alpha}{n}$, where λ is taken between the limits 0 and $\frac{2n-1+(-1)^n}{4}$

X. If an equilateral triangle have its angular points in 3 parallel straight lines of which the middle one is distant p from the outside ones by a angle, its side will be $2\sqrt{a^2+ab+b^2}$

XI. At each end of a horizontal base measured in a known direction from the place of an observer, the angle which the distance of the other end and a certain object subtends is observed, as also the angle of elevation of the object: find its height and bearing

XII. Investigate the following expressions for $\cos n\theta$, n being an even integer:

$$\cos n\theta = 1 - \frac{n^2}{2}\sin^2\theta + \frac{n^2(n^2-2^2)}{4}\sin^4\theta - \frac{n^2(n^2-2^2)(n^2-4^2)}{6}\sin^6\theta + \&c$$

$$\cos n\theta = 2^{n-1}\left\{\sin^2\alpha - \sin^2\theta\right\}\cdot\left\{\sin^2 3\alpha - \sin^2\theta\right\}\&c - \left\{\frac{1}{2}\right\}\left\{\sin^2(n-1)\alpha - \sin^2\theta\right\}$$

when $n\alpha = \pi$

Trigonometrical Problems

Dec. 1835.

Find the values of x in the eq^n: $2\sqrt{x^2-x''} = \sin A$.

Prove the following formulæ

$\operatorname{cosec} A = \sqrt{1+\cot^2 A}$, $\cos 2A + \cos 2B = 2\cos(A+B)\cos(A-B)$; $\tan\left(\frac{\pi}{4}+\frac{A}{2}\right) = \sqrt{\frac{1+\sin A}{1-\sin A}}$.

and others that if $A+B+C = 2s$

$\cos 2s + \cos 2(s-A) + \cos 2(s-B) + \cos(2(s-C) = 4\cos A \cos B \cos C$.

Find the value of $\tan 36^\circ$ to four places of decimals.

If $a.b.c$ be the sides of a triangle, of which the angle $C = 90^\circ$ shew that

$\tan\frac{A-B}{2} = \frac{a-b}{a+b}$, and $\cos(2A-B) = \frac{a}{c^2}(3c^2-4a^2)$

Prove also that $A = \frac{1}{2\sqrt{-1}} \log \frac{b+a\sqrt{-1}}{b-a\sqrt{-1}}$

The area of a regular pentagon: area of an isosceles triangle used in its construction (See B (V.. XIV)) as $\sqrt{5}:1$ —

The sides of a triangle are as the numbers 4.5.6, find the $\angle$s

given $\log_{10} 2.3010299$ $\quad$ $\log_{10} 5.6989700$

$\log \cos 27^\circ 53' = 9.9464040$

$\log \cos 27^\circ 54' = 9.9463371$

Explain the use of subsidiary angles; shew that the eq^n $\cos(a+x) = \cos x \sin x + \sin a$, may be reduced to the form

$\cos(\theta+x) = \frac{\sin a}{\cos \alpha} \cdot \cos\theta$, $\tan\theta = \frac{\sin(45+a)}{\cos 45 \cos a}$

If $\cos(a-x) = \cos x \cosh$, where x is very small prove that

$x = -2\cot a \sin^2\frac{b}{2}\left(1-\cot^2 a \sin^2\frac{b}{2}\right)$ very nearly —

How is the number of seconds contained in x to be found from this formula.

If n is a regular polygon of n sides straight lines be drawn from the extremities of a side (a) to those of any other side so as to cross each other shew that the locus of their intersection is a circle whose rad$^s = \frac{a}{2}\operatorname{cosec}\frac{2\pi}{n}$

A person wishing to determine the length of a inaccessible line suitably places himself at two P.s of observ^n, a.d then due to of the objects, at such distances that the $\angle$ which the small pattens at the two positions each $= a$: prove that if a be the distance between the two stations, the breadth of the vale $= a\tan\frac{a}{2}$.

If in the three edges which meet at one angle of a tetrahedron A.B.C be taken at distances $a.b.c$ from the angle respectively, the area of the triangle formed by joining these points with each other $= \frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$.

If θ be a small $\angle$ contain^g n ", prove the following formulæ and explain the use — $\log_{10} n = \log\sin n'' + \frac{1}{3}(10-\log\cos\theta) - \log\sin n''$

Conic Sections

Exercises

Find in terms of their coordinates the length of the line joining two points, and its inclination to the axis of x.

Having given the eqns to two straight lines find the angle between them (I) when the axes are rectangular (2) when they are oblique.

Investigate the equation to a straight line, and reduce it to the forms $\frac{x}{a}+\frac{y}{c}=1$ $x\cos\alpha+y\sin\alpha=p$; $r=p\sec(\theta-\alpha)$.

Explain the meaning and object of transformation of co-ordinates. Shew how to pass from one system of oblique co-ordinates to another. and adapt the formulæ (I) to the case when both systems are rectangular (2) when the first only is rectangular.

Find the polar eqn to a circle; and determine the radius and co-ordinates of the centre of the circle $x^2+y^2+6ax-4ay-3a^2=0$

Define the centre and axis of a curve, and shew that the origin is the centre and the axes of the co-ord. are axes of the curve $ax^2+2hxy+by^2=c$

If a tangent to a parabola at a point P meet the directrix in Q and a tangent at the vertex in Y QSP, PYS will be right angles.

In the ellipse shew that the describing ~~focal~~ point must be on the same side of the directrix as the focus; and having given the eccentricity and major axis, find the minor axis, and the latus rectum and the distances of the focus and directrix from the centre.

Find the distance of any point in the ellipse from the foci; and shew that the foci are the only points the distances of which from every point in the ellipse can be expressed rationally in terms of the co-ordinates.

In the ellipse prove that the tangent at any point bisects the external between the focal distances, and thence draw tangents to an ellipse from a given external point.

Express the length of the perpendicular from the centre on the tangent at any point of an ellipse in terms of its inclination to the major axis; find the equation to the locus of the extremity of the perpendicular and to the locus of the intersection of two tangents which cut at right angles.

Find the locus of a point the difference of whose distance from two given points is invariable, and shew how the curve may be described by a continuous motion.

Find the polar eq^n of the hyperbola the external focus being pole the curve, and shew that the eq^n will represent both branches.

Having given the eq^n to an hyperbola referred to its axes, find the eq^n to a line touching it.

Trigonometrical Problems.

XIII. Shew that a large error may be expected in determining $\tan\theta$ from the formula $\cot\theta - \tan\theta = 2\left\{\sqrt{\frac{1+\cos 4\theta}{1-\cos 4\theta}}\right\}$ when θ is small, or nearly 45° or nearly 90° and find the error corresponding to a given tabular error when $\theta = 30^\circ$

XIV. If $\delta, \delta', \delta'', \delta'''$ be the distances of the circumscribing circle whose rad $= R$ from the centres of the inscribed and escribed circles whose radii are r, r', r'', r''' shew that $12R^2 = \delta^2 + \delta'^2 + \delta''^2 + \delta'''^2$ and $4R + r = r' + r'' + r'''$

Theory of Eqns. Dec. 37.

XI. Shew that every rational symmetrical function of the roots of a eqn can be expressed by its coefficients and find the value of $\Sigma(a^m b^k)$ when a, b are roots of $x^2 - 1 = 0$. Also find the sum of the 8th powers of the roots of $x^3 + 3x + 2 = 0$

XII. Apply Sturm's method to shew that $x^3 + 2x^2 - 3x + 2 = 0$ has only one real root, and that it lies between -1 and -10.

XIII. Determine all the systems of values of x and y which will satisfy the eqns $x^3 - 3yx^2 + (3y^2 - y + 1)x - y^3 + y^2 - 2y = 0$ and

$$x^2 - 2yx + y^2 - y = 0$$

4 o show that $(a+b+c)^3 > 24abc$ & $< 9(a^3+b^3+c^3)$

by actual division

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-ac-bc)$$

(1) $$= \tfrac{1}{2}\{(a-b)^2+(a-c)^2+(b-c)^2\}(a+b+c)$$

$$a^3+b^3+c^3+6abc = (a+b+c)^3 - 3\{a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)\}$$

$$= (a+b+c)^3 - 3\{c(a-b)^2+b(a-c)^2+a(b-c)^2\} \div 18abc$$

add $18abc$ to both sides

(2) $$a^3+b^3+c^3+24abc = (a+b+c)^3 - 3\{c(a-b)^2+b(a-c)^2+a(b-c)^2\}$$

By successively eliminating $a^3+b^3+c^3$ & abc between Eq^{ns} (1) & (2)

we find $27abc = (a+b+c)^3 - \frac{1}{2}\{\overline{a+b+7c}(a-b)^2+\overline{a+c+7b}(a-c)^2+\overline{b+c+7a}(b-c)^2$

which shows that $27abc < (a+b+c)^3$

again $9(a^3+b^3+c^3) = (a+b+c)^3 + \overline{4a+4b+c}(a-b)^2 + \overline{4a+4c+b}(a-c)^2$

$$+ \overline{4b+4c+a}(b-c)^2$$

which will prove &c

It to reduce x^2+px+q into 2 factors of the 1st degree x when any quantity different from the roots is written for x in the equation $x^2+px+q=0$ show in what cases the result will be $+\alpha-\beta$ provided that it can never be negative so long as p is $> q$

Let $x^2+px+q = x^2+px+\left(\frac{p}{2}\right)^2 - m^2$ when $p^2-\frac{4m^2}{4} = q$

$\therefore \left(x+\frac{p}{2}\right)^2 - m^2 = \left\{\left(x+\frac{p}{2}\right)-m\right\}\left\{\left(x+\frac{p}{2}\right)-m\right\} = (x-\alpha)(x-\beta)$ when

$\alpha = -\left(\frac{p}{2}+m\right)$ $\beta = -\left(\frac{p}{2}-m\right)$

It follows that x^2+px+q may be reduced to nothing by writing either α or β for x & in no other way for by substituting α or β for x we reduce the 2d member of Eq. to nothing & ~~it~~ must reduce the first to nothing for the 1st is identical with the 2d

If any quantity different from α and β were substituted for either factor would be reduced to 0

$\therefore$ their product would not $=0$

$\therefore$ 1st member would not $=0$

by the substituting of any quantity different from α or β

It also follows since $x^2+px+q = (x-\alpha)(x-\beta)$ that if any quantity $> \alpha$ or β be substituted for x the result will be positive if intermediate between α and β negative

The solution of 2 equations of the 2d degree between 2 unknown quantities depends on the solution of an Eq. of the 4th degree with respect to an unknown quantity of a degree not greater than ~~4~~ the 4th.

For in the Eq. general of the 2d degree

$$\left.\begin{aligned} ay^2+bxy+cx^2+dy+ex+f &= 0 \\ a'y^2+b'xy+c'x^2+d'y+e'x+f' &= 0 \end{aligned}\right\} \text{ by eliminating } x^2 \text{ & equating results}$$

$$c'(ay^2+bxy+dy+ex+f) = c(a'y^2+b'xy+d'y+e'x+f')$$

$$x = \frac{(ca'-ac')y^2+(c'd-cd')y+cf'-fc'}{(bc'-cb')y+cc'-cc'}$$

since the numerator of this expression is of the 2d degree in y, its square is the 4th degree & when the expression is substituted for x in either of the given Eq

Let a be the given base ax the other side ay the hypothenuse then to find the rational values of x and y which satisfy the eq^n $y^2 - x^2 = 1$ $\therefore$ $(y+x)(y-x) = 1$ the only quantities of which is reciprocal to the other assume $y + x = n$ $\therefore$ $y - x = \frac{1}{n}$ $\therefore$ $ax = \frac{a}{2}\left(n - \frac{1}{n}\right)$ $ay = \frac{a}{2}\left(\frac{n^2+1}{n}\right)$

Req^d the series of fractions converging to n in the eq^n $2^n = 6$

It is manifest that n lies between 2 & 3 ~~let $n = \frac{a}{b} + \frac{1}{x}$~~

$\therefore$ $n = 2 + \frac{1}{x'}$ $2^{x'} = \frac{6}{4} = \frac{3}{2}$ $2 = \left(\frac{3}{2}\right)^{x'}$ $x' = 1 + \frac{1}{x''}$

The quotients are 2. 1. 1. 2. 2 &c

convergents are $\frac{2}{1}$ $\frac{3}{1}$ $\frac{5}{2}$ $\frac{13}{5}$ $\frac{31}{12}$ & the diff bet: n & $\frac{31}{12} < \frac{1}{144}$

If $\frac{a}{b} = \frac{c}{d}$ then will $a + d = b + c + \frac{(a-b)(a-c)}{a}$

& $\frac{1}{a} + \frac{1}{d} = \frac{1}{b} + \frac{1}{c} + \frac{(a-b)(a-c)}{a}$

Since $a + d = b + c + x$ x being unknown

$\therefore$ $x = a + d - b - c$ ~~$d = \frac{bc}{a}$~~

$\therefore$ $x = a + \frac{bc}{a} - b - c$

$\therefore$ $a + d = b + c + \frac{a^2 + bc - ab - ac}{a} = b + c + \frac{(a-b)(a-c)}{a}$

& dividing by $ad = bc$

$\frac{1}{a} + \frac{1}{d} = \frac{1}{b} + \frac{1}{c} + \frac{(a-b)(a-c)}{abc}$

& perimeters of circular polygons being assumed to be proportional to their homologous sides and their areas to the squares of them to prove that the circumferences and areas of circles are respectively proportional to their radii & the squares of their radii

Let C c be the circumferences of the circles R r their radii P p the perimeters of 2 similar polygons let $X = C - P$ $x = c - p$ then whatever be the no. of the sides of the polygons provided they are similar we shall have

$$\frac{R}{r} = \frac{P}{p} = \frac{C - X}{c - x} \quad \therefore \quad \frac{C}{R} = \frac{c}{r} = \frac{X}{R} - \frac{x}{r}$$ now

since C c R r are invariable if $\frac{C}{R}$ be not $= \frac{c}{r}$ their difference is an invariable finite quantity let this be k then $\frac{X}{R} - \frac{x}{r} = k$ but by increasing the No. of sides of the polygons we may make X & x $\frac{X}{R} - \frac{x}{r}$ as small as we please $\therefore$ we may make it less than the assigned quantity k $\therefore$ our assumption is absurd & $\frac{C}{R}$ is & unequal $\frac{c}{r}$ that is they are equal

Recurring series

When the general term of a series is known the sum of the No. of terms may be known by regarding it as made up of the general terms of 2 or more known series

If $Rx^{n-2} + Sx^{n-1} + Tx^n$ be 3 consecutive terms of quotient of $\frac{a + bx}{1 + \alpha x + \beta x^2}$ &

have $a + bx = 1 + \alpha x + \beta x^2 \{ \text{---} \; \text{&c} \; + Rx^{n-2} + Sx^{n-1} + Tx^n$

$\therefore$ multiplying this out & equating coefficients of x^n we find

$$T + S\alpha + R\beta = 0$$

Hence if this be the general relation between the coefficients of 3 consecutive terms of a series ascending by powers of x we know that the series has arisen from the development of a fraction whose denominator is $1 + \alpha x + \beta x^2$

Similarly it may be shown if the general relation

$$T + S\alpha + R\beta + \text{&c} \; \text{---} + L\lambda = 0$$ the series has arisen from the development of a fraction whose denominator is

$$1 + \alpha x + \beta x^2 + \text{&c} \; + \lambda x^{\lambda - 1}$$

Then the cube root consists of $2n+2$ digits and $n+2$ can be found by the ordinary rule the rest can be found by dividing by the trial divisor Let $a+b$ be the cube root where a consists of $n+2$ digits & b consists of n digits

$a^3 + 3a^2b + 3ab^2 + b^3$ the quantity whose root is required

Then after a has been found we have remainder $= 3a^2b + 3ab^2 + b^3$

the trial divisor $= 3a^2$

$\therefore$ quotient $= b + \frac{b^2}{a} + \frac{b^3}{3a^2}$ If this be adopted as the nine $\&$ b, error $= \frac{b^2}{a} + \frac{b^3}{3a^2}$ Now $b < 10^n$ $\therefore b^2 < 10^{2n}$

a is not $< 10^{2n+1}$ $\therefore \frac{b^2}{a} < \frac{10^{2n}}{10^{2n+1}} < \frac{1}{10}$ also $b^3 < 10^{3n}$

a^2 is not $< 10^{4n+2}$ $\therefore \frac{b^3}{a^2} < \frac{1}{3\times 10^{n+2}}$ $\therefore$ error $< \left(\frac{1}{10} + \frac{1}{3\times 10^{n+2}}\right) < 1$

that is the last or figures of the root may be obtained by division only

Solve the equation $ax + b = cx + d$ & shew that it has only one root - How do you interpret the result when negative when of form $\frac{0}{0}$ and when $\frac{\text{finite quantity}}{0}$

$$ax + b = cx + d$$
$$ax - cx = d - b$$
$$x = \frac{d-b}{a-c}$$

If it has ~~any~~ two roots let them be p and q by substitution

$$ap - cp = d - b$$
$$\underline{aq - cq = d - b}$$
$$a(p-q) - c(p-q) = 0$$
$$(a-c)(p-q) = 0$$

but $a - c$ is not $= 0$ for then the equation is no equation at all $\therefore p - q = 0$ or $p = q$ that is p and q have not different values & there is only one value of x which satisfies the equation

Suppose $d = b$ and $a = c$ then $x = \frac{0}{0}$ & its value is indeterminate

Suppose d not $= b$ but $a = c$ $\therefore ax + b = ax + d$ & $b = d$ but b is not $= d$ $\therefore$ the problem is absurd If the answer be of the form $-\frac{a}{b}$ the problem is also absurd

Solve the equation $\frac{\sqrt{1+x}+\sqrt{x-1}}{\sqrt{1+x}-\sqrt{x-1}} = 2x$ and prove that whatever value be assigned to x $x \pm 1$ is a square number.

by formula from Euclid if $\frac{a}{b} = \frac{c}{d}$ $\frac{a \pm b}{a \mp b} = \frac{c \pm d}{c \mp d}$

$$\frac{\sqrt{x-1}}{\sqrt{x+1}} = \frac{2x^2-1}{2x^2+1}$$

$$\frac{x-1}{x+1} = \left(\frac{2x^2-1}{2x^2+1}\right)^2$$

$$(x-1)(2x^2+1)^2 = (x+1)(2x^2-1)^2$$

$$4x^2x - 4x^4 - 1 = 4x^4 - 4x^2x + 1$$

$$8x^2x = 8x^4 + 2$$

$$x = x^2 + \frac{1}{4x^2}$$

$$x \pm 1 = x^2 + \frac{1}{4x^2} \pm 1 = \left(2x^2 \pm 1\right)^2$$

$\therefore$ whatever value you assign to x $\left(2x^2 \pm 1\right)^2$ is a square number

$\therefore$ $x \pm 1$ is a S.D.

Given $a'x + b'y + c'z = d'$ (1)
$a''x + b''y + c''z = d''$ (2)
$a'''x + b'''y + c'''z = d'''$ (3)
} required x y z

Multiply (2) by m (3) by n and add (1) to the resulting equation
Then we have

$$(a' + ma'' + na''')x + (b' + mb'' + nb''')y + (c' + mc'' + nc''')z$$
$$= d' + md'' + nd'''$$

Now to find x let the arbitrary multipliers m and n be such that the coefficients of y and z in this last equation are separately equal to 0 that is

$$\left.\begin{aligned} b' + mb'' + nb''' &= 0 \\ c' + mc'' + nc''' &= 0 \end{aligned}\right\}$$

$$\text{or} \quad \left.\begin{aligned} b'c''' + mb''c''' + nb'''c''' &= 0 \\ b'''c' + mb'''c'' + nb''c''' &= 0 \end{aligned}\right\}$$

$$\therefore b'c''' - b'''c' + (b''c''' - b'''c'')m = 0$$

$$\text{or} \quad m = \frac{b'''c' - b'c'''}{b''c''' - b'''c''}$$

similarly $n = \dfrac{b'c'' - b''c'}{b''c''' - b'''c''}$

Then we have x

$$= \frac{d' + md'' + nd'''}{a' + ma'' + na'''}$$ in which the above values may be substituted for m and n —

Similarly by making the coefficients of x and z or of x and y separately equal to 0 the value of y or of z may be found — Hence the following rule may be deduced and found easy of application :—

To find x multiply the 1st Eq: by $b''c''' - b'''c''$ the 2nd by $b'''c' - b'c'''$ and the 3rd by $b'c'' - b''c'$ then add together the resulting equations and a simple Equation will be obtained in which y and z do not appear — Having found x its value being substituted y and z may be found according to the usual rules

Express $(a+b+c+d+e+f)^2$ 1° in terms of $a^2+b^2+c^2+d^2+e^2+f^2$ and $\overline{a-b}^2+(a-c)^2+$
2° in terms $(a+b)^2+(a+c)^2$ &c in the squares & sums of squares of every 2
3° in terms of $(a+b)^2+(a+c)^2+(b+c)^2+$ &c $(a-b)^2+(a-c)^2+$ &c
4° in terms of $ab+ac+bc+$ &c $(a-b)^2+(a-c)^2$ &c

Let $P = a^2+b^2+c^2+d^2+$ &c
$Q = ab+bc+ce+$ &c
$R = (a+b)^2+(a+e)^2+(b+c)^2+$ &c
$S = (a-b)^2+(a-c)^2+(b-c)^2$ &c

Then $(a+b+c+d+e+f)^2 = nP - S$ (1)
$= R-(n-2)P$ (2)
$= \frac{nR-(n-2)S}{2n-2}$ (3)
$= \frac{2nQ+S}{n-1}$

To show that $\frac{ma+nb}{m+n}$ lies between a & b

$$\frac{ma+nb}{m+n} = a - \frac{n}{m+n}(a-b)$$
$$= b + \frac{m}{m+n}(a-b)$$

To show that $\frac{ax+by}{ax^2+by^2}$ lies bet: $\frac{1}{x}$ & $\frac{1}{y}$

$$\frac{ax+by}{ax^2+by^2} = \frac{1}{x} - \frac{by^2}{ax^2+by^2}\left(\frac{1}{x}-\frac{1}{y}\right)$$
$$= \frac{1}{y} + \frac{ax^2}{ax^2+by^2}\left(\frac{1}{x}-\frac{1}{y}\right)$$

which proves &c

Find the least common integral multiple of $2\frac{1}{2}$ $6\frac{1}{3}$ and 5

Since 5 is a multiple of $2\frac{1}{2}$ we need only find the L.C.M. of $6\frac{1}{3}$ and 5

These reduced to a common denominator are $\frac{19}{3}$ and $\frac{15}{3}$.

$\frac{1}{3} \mid \frac{19}{3} \cdot \frac{15}{3}$

$19 \cdot 15$ and 19 & 15 have no common measure

$\therefore \frac{1}{3} \times 19 \times 15 = 19 \times 5 = 95$ is the L.C.M. of the quantities proposed

Find the greatest fraction by which $\frac{2}{9}$ $\frac{1}{3}$ $\frac{5}{8}$ are integral multiples

The least comm. denom. is 72

$\therefore \frac{1}{72}$ is the fraction req?

Req? the relation of a b á b' in order that x^2+ax+b and $x^2+áx+b$ may have a common measure and when they have find it

Let α be a common measure of the 2 quantities ~~and~~ that is that a is cont? in x^2+ax+b p times and in $x^2+á x+b'$ q times

$\therefore \frac{x^2+ax+b}{p} \cdot \frac{x^2+áx+b'}{q}$

$q(x^2+ax+b) = p(x^2+á x+b')$

Let $q = p$

$\therefore x^2+ax+b = x^2+áx+b'$

$ax+b = á x+b'$

Now let $á = a$ $b' = b$ then $ab = áb'$ which is true

The 2 quantities have a common measure & their G.C.M. will be x^2+ax+b

Show that no numerical value of y will make $yx^3-(y^3-3y-1)$ and x^2+y^2+3 have a common measure

$$x^2-y^2+3 \;\big)\; x^5y - xy^3 + 3xy + x + y \;\big(\; xy$$
$$x^5y - xy^3 + 3xy$$
$$x+y \;\big)\; x^2-y^2+3 \;\big(\; x-y$$
$$x^2+xy$$
$$-xy-y^2+3$$
$$-xy-y^2$$
$$3$$

Since the rem: is numerical & no value of y whatever in it & no numerical value of y can give them a common measure unless y in the rem: in any way ~~was integer~~ it might be put = 0 or some integer & thus the rem: would be destroyed but as is not the prop. is proved –

If a is that number whose cube is nearest to a given number then the cube root of n may be taken approximately to be equal to $a\left\{\frac{2n+a^3}{n+2a^3}\right\}$.

Since cube root of n differs very little from a we may assume $\sqrt[3]{n} = a+x$ where x is small when compared with a then by substituting $(a+x)^3$ for n in the above formula, leaving out x we have

$$\frac{2(x+a)^3+a^3}{(x+a)^3+2a^3}$$

and dividing by a^3 we have

$$\frac{2\left(1+\frac{x}{a}\right)^3+1}{\left(1+\frac{x}{a}\right)^3+2} = \frac{3+\frac{6x}{a}}{3+3\frac{x}{a}} = \frac{1+\frac{2x}{a}}{1+\frac{x}{a}} = 1+\frac{x}{a}$$

neglecting $\left(\frac{x}{a}\right)^2$

$$\therefore \sqrt[3]{n} = a+x = a\left\{\frac{2n+a^3}{n+2a^3}\right\}$$

See other page

A different and more easy way

Let $\sqrt[3]{N} = a + x$ when x is small when compared with a

$\therefore N = a^3 + 3a^2x + 3ax + x^3$

$\therefore N - a^3 = 3a^2x$ nearly

$$x = \frac{N - a^3}{3a^2} \text{ nearly}$$

$$\therefore N - a^3 = \left\{3a^2 + 3a\left(\frac{N-a^3}{3a^2}\right)\right\} x \text{ nearly}$$

$$= \left\{3a^2 + \frac{N-a^3}{a}\right\} x \text{ nearly}$$

$$\therefore x = \frac{N - a^3}{3a^2 + \frac{N-a^3}{a}}$$

$$\therefore \sqrt[3]{N} = a + x = a + \frac{N-a^3}{3a^2 + \frac{N-a^3}{a}} = a + \frac{N-a^3}{2a^3+N} = \frac{2a^4 + aN + aN - a^4}{2a^3 + N}$$

$$= \frac{a^4 + 2aN}{2a^3 + N} = a\left\{\frac{2N + a^3}{2a^3 + N}\right\}$$

Q. E. D.

Equations by Dr Thomson. See Lewis appendix

$$(1-x^2)^2(1+y^2) - (1+x^2)(1-y^2) = 4x^2(1+y^4)^{\frac{1}{2}}$$

$$4xy = \sqrt{2}(1-x^2)(1-y^2)$$

$$\left(\frac{1-x^2}{1+x^2}\right)^2 - \frac{1-y^2}{1+y^2} = \frac{4x^2}{(1+x^2)^2}\sqrt{\left\{\frac{1+y^4}{(1+y^2)^2}\right.}$$

$$\frac{4x^2}{(1-x^2)^2}\cdot\frac{4y^2}{(1-y^2)^2} = 2$$

$$\text{or } x^2 v = (1+x^2)\sqrt{\frac{1+v^2}{2}}$$

$$\left(\frac{1}{x^2}-1\right)\left(\frac{1}{v^2}-1\right) = 2$$

$$\therefore \frac{1}{x^2}-1 = \frac{2v^2}{1-v^2} \quad \text{or} \quad \frac{1}{x^2} = \frac{1+v^2}{1-v^2}$$

$$1 - \frac{v+v^3}{1-v^2} = \frac{v^2}{1-v}\sqrt{2+2v^2}$$

$$1-v^2-v(1+v^2) = v^2\sqrt{2+2v^2}$$

$$(1-v^2)^2 - 2v(1-v^4) + v^2(1+v)^2 = 2v^4(1+v^2)$$

$$\text{or } (1-v^2)\{1-2v(1+v^2)+v^4\} = 0$$

$$\therefore v = 1 \text{ or } -1$$

$$v^4+1 = 2v(1+v^2) = 0$$

$$\text{or } \left(v+\tfrac{1}{2}\right)^2 - 2\left(v+\tfrac{1}{2}\right)+1 = 3$$

$$v+\tfrac{1}{2} = \pm\sqrt{3}+1$$

$$v = \pm\sqrt{3}+\tfrac{1}{2}.$$

Plane trigonometry.

1) Having given two sides a. b. and ∠ A opposite one of them solve the triangle.

2) Solve a triangle from having the three sides given.

3) Show how to find the height of an object whose foot is inaccessible

4) Given the three sides of a triangle to find the radii of the circles circumscribed about, and inscribed in it

5) To find the area of a polygon of n sides having its sides given thence deduce the area of a circle.

6) Show how to find an angle from a log sine which is not given in the table.

Ex. log sin 6°. 32'. 30" = 9.0366218 } find angle whose log sine =
log sine 6°. 32'. 40" = 9.0368054 } 9.0367650

7) Determine A from the eqn. tan A = a where a is very large.

8) What are the cases in which the solution of a triangle from three parts is impossible, and when ambiguous. Also state when a triangle is impossible.

9) What difference is there between the limits of any ∠ in Geometry and Trigonometry?

10. Express sin $\frac{A}{2}$ in terms of sin A and hence show why the result gives 4 values. How do you discriminate between these values.

11. Inscribe an angle, and find the tan of half an angle in terms of the tangent of the whole angle.

12. Prove fully. $\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$

$\cos A = \frac{1}{1 + \tan A \tan \frac{A}{2}}$.

Exercise 61. Equations.

$$\left.\begin{aligned}(x^2+y^2)xy &= 78\\ x^4+y^4 &= 97\end{aligned}\right\}$$

From 1. $x^3y+xy^3 = 78$

$4x^3y+4xy^3 = 312$ (A)

Also from (1). $x^2+y^2 = \frac{78}{xy}$ (B)

From the (11) $x^4+2x^2y^2+y^4 = 97+2x^2y^2$

$x^2+y^2 = \sqrt{97+2x^2y^2}$ (C)

Equate B and C.

$$\frac{78}{xy} = \sqrt{97+2x^2y^2}$$

$$\frac{6084}{x^2y^2} = 97+2xy$$

$$x^4y^4+\frac{97}{2}x^2y^2+\frac{9409}{16} = \frac{58081}{16}$$

$$x^2y^2 = 36$$

$$6x^2y^2 = 216 \text{ (D)}.$$

Add together (11). (A) and (D)

$x^4+4x^3y+6x^2y^2+4xy^3+y^4 = 625$

$x+y = 5$

Also $x^4+y^4 = 97$

$x^4+2x^2y^2+y^4 = 169$

$x^2+y^2 = 13$

again $x+y = 5$

$x^2+2xy+y^2 = 25$

$4xy = 24$

$x^2-2xy+y^2 = 1$

$x-y = \pm 1$

$x+y = 5$

$2x = 6$

$x = 3$

$2y = -4$

$y = -2.$

or $x-y = -1$

$x+y = 5$

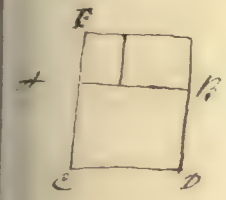
$2x = 4$

$x = 2$

$2y = 6$

$y = 3$

A parallelopiped containing 19 feet is divided into 2 smaller figures, the smallest of which is a cube. The content of the largest and smallest together is to the content of the other piece as 5 to 1. and the largest piece has its two greatest faces squares. what are the dimensions of the parallelopiped?



Let AC = x and the depth of the parall. = z

$\therefore xz$ = content of larger piece.

z^3 = do —— smaller piece

$\frac{(x^2+z^2)x}{5}$ = do —— small

$(x+z)xz$ = whole content = 12

x^2z = do —— larger

xz^2 = content of the two smaller pieces

$xz^2 - z^3 = (x-z)z^2 = 2$. $\therefore x = \frac{2+z^3}{z^2}$ (I).

Also $x^2z + z^3 = 10$ $\therefore x^2 = \frac{10-z^3}{z}$ or $x = \sqrt{\frac{10-z^3}{z}}$ (II)

Equate I and II. $z^6 - 3z^3 = -2$ or $z = 1$

Also $x = \frac{2+z^3}{z^2} = 3$

and $CE = x + z = 3 + 1 = 4$

∴ the length of the parallelopiped = 4 feet

breadth — — 3 feet

depth 1 foot

and the several parts contain the first 4 feet

second — 3 feet

third 1 foot

12 feet

Let x = time before 12 that A commenced

$x - \frac{1}{2}$ = do ——— B ———

$\therefore x + 8\frac{3}{4} = \frac{4x+35}{4}$ = whole time that A was engaged

$x + 5\frac{1}{2} = \frac{2x+11}{2}$ = do ——— B ———

$$\therefore \frac{6 \times 2}{2x+11} + \frac{8\frac{3}{4} \times 4}{4x+35} = 1$$

$$12 + \frac{70x + 385}{4x+35} = \frac{2x+11}{2}$$

$$48x + 420 + 70x + 385 = 8x^2 + 70x + 44x + 385$$

$$8x^2 - 4x = 420$$

$$x^2 - \frac{1}{2}x = \frac{105}{2}$$

$$x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{841}{16}$$

$$x - \frac{1}{4} = \frac{29}{4}$$

$$x = \frac{30}{4} = 7\frac{1}{2}$$

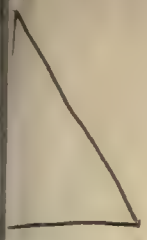
$$\therefore 12 - 7\frac{1}{2} = 4\frac{1}{2}$$

$$12 - 7 = 5$$

or A began at ½ past 4

B ——— 5

Luis Cabanes d - Sec Colegio - Jerada. Lyon.



Let $AC = x$, $BC = y$ $\therefore AB = \sqrt{x^2+y^2}$

$x + \frac{1}{2}\sqrt{x^2+y^2} : y + \frac{1}{2}\sqrt{x^2+y^2} :: 13 : 11$

$\therefore 11x - 13y = \sqrt{x^2+y^2}$ (I)

Also $y + 30 + \frac{1}{2}\sqrt{x^2+y^2} : x - 30 + \frac{1}{2}\sqrt{x^2+y^2} :: 13 : 11$

$13x - 11y + \sqrt{x^2+y^2} = 720$

Equate I and II $\qquad \sqrt{x^2+y^2} = 720 - 13x + 11y$ (II)

$11x - 13y = 720 - 13x + 11y$

$24x = 24y + 720$

$x = y + 30$. (A)

Again $y + 30 + \frac{1}{2}\sqrt{2y^2+60y+900} : y + \frac{1}{2}\sqrt{2y^2+60y+900} :: 13 : 11$

$330 - 2y = \sqrt{2y^2+60y+900}$

Square and transpose

$y^2 - 690y = -54000$

$y^2 - 690y + 119025 = 65025$

$y - 345 = \pm 255$

$y = 345 - 255 = 90$

$x = y + 30 = 90 + 30 = 120$

$\sqrt{(x^2+y^2)} = \sqrt{8100 + 14400} = \sqrt{22500} = 150$

$$a = \sqrt{\frac{b^3}{c^3}v + \frac{b^6(v+1)^2}{4c^6} + \frac{a^3}{2c^3}(v+1)}$$

$$= \sqrt[3]{\left(v + \frac{b^3}{} - \right.} \quad - \quad .$$

Two boys whose speed are in the proportion of 13 to 11. Start off from the right angle of a right angled field — they meet in the middle of the opposite side, and again the distance of 30 yards from the place where they started — find the length of the sides —

Indeterminate Equations.

Bacchus and Silenus agreed by the side of a full cask, and resolved to spend the opportunity of drinking, which he continued for two thirds of the time that Silenus would have taken to empty the whole cask. After that Silenus awoke, and drank at the cask what had left. Had they drunk both together, it would have been emptied two hours sooner and Bacchus would have drunk one half of what he left Silenus. Required the time in which they would empty the cask seperately. —

Let x = time in which Bacchus would empty the cask
y = " " " Silenus drink ——— x —
$\therefore \frac{2y}{3}$ = time that Silenus was asleep
$1 \frac{2y}{3x}$ = part of the whole that Bacchus drank during that time.
$1 - \frac{2y}{3x} = \frac{3x-2y}{3x}$ = part that was left for Silenus to drink.
$\frac{3xy-2y^2}{3x}$ = time that Silenus was drinking.
$\therefore \frac{2y}{3} + \frac{3xy-2y^2}{3x} = \frac{5xy-2y^2}{3x}$ = { the whole time that Bacchus and Silenus were drinking.

Also $\frac{1}{x} + \frac{1}{y} = \frac{y+x}{yx}$ = part they would both drink together
$\therefore \frac{yx}{y+x}$ = { time they would take to empty the cask if both drank together.

$\therefore \frac{5xy-2y^2}{3x} = \frac{yx}{y+x} + 2$ (A) and $\frac{2y}{y+x} = \frac{3x-2y}{3x} = B$.

From B we have $3x^2 - 5yx = 2y^2$

$$\therefore x^2 - \frac{5y}{3}x + \frac{25y^2}{36} = 2y^2 + \frac{25y^2}{36} = \frac{49y^2}{36}$$

$$\text{or} \quad x = 2y$$

and substituting this value of x in (A) we have

$$\frac{10y^2 - 2y^2}{6y} = \frac{2y^2 + 2y + 4y}{3y}$$

$$24y = 12y + 36$$

$y = 3$ = the time that Silenus would require to empty the cask.

and $x = 2y = 6$ = the time that Bacchus would require to empty the cask. —

$x-4\sqrt{3}\,,\ (x-2\sqrt{3})(x-6\sqrt{3}) = x^3 - 7\sqrt{3}x^2 + 42x - 24\sqrt{3}$

Here $u_1 + u_2 = 3hq - 9r = 666\sqrt{3}$

$u_1 u_2 = 9q^3 + 9(h_3 - 6hq)r + 81r^2$

$= 666792 - 476280 + 139968 = 330480$

$u_1 - u_2 = 54\sqrt{3}$

$u_1 = (333 + 27)\sqrt{3} = 360\sqrt{3}$

$u_2 = (333 - 27)\sqrt{3} = 306\sqrt{3}$

$u_3 = 1029\sqrt{3} - 666\sqrt{3} = 363\sqrt{3}$

$z_1 = 363\sqrt{3} + 180\{-1 + \sqrt{-3}\}\sqrt{3} + 153\{-1 - \sqrt{-3}\}$

$= \{363 - 333\}\sqrt{3} + 27\sqrt{-3}$

$= 30\sqrt{3} + 27\sqrt{3}\sqrt{-3}$

$z_2 = 30\sqrt{3} - 27\sqrt{3}\sqrt{-3}$

$\sqrt[3]{z_1} = \sqrt[3]{30\sqrt{3}}\sqrt[3]{1 + \frac{9}{10}\sqrt{-3}}$

$\sqrt[3]{10}\cdot\sqrt{3}\{1 + \frac{3}{10}\sqrt{-3} + \frac{1}{9}\cdot\frac{243}{1000}\}$

$= \sqrt[3]{10}\sqrt{3}\cdot\{1.027 + \frac{3}{10}\sqrt{-3}\ \alpha\ \text{—}$

$\sqrt[3]{z_2} = \sqrt[3]{10}\sqrt{3}\{1.027 - \frac{3}{10}\sqrt{-3}\ \text{— —}$

$a = \frac{1}{3}\{7\sqrt{3} + (2\sqrt[3]{10}\sqrt{3} \times 1.027)$

$\frac{1}{3}\{7 + 2.466\}\sqrt{3} = \frac{1}{3}\{11.416\sqrt{3} = 4\sqrt{3}$ nearly.

Then the proposed equation divided by $x - 4\sqrt{3}$ will be reduced to a quadratic and the real roots will be found.

Cubic Equations

$(x+6)(x+3)(x+5) = x^3 + 14x^2 + 63x + 90$ —

Here $p = -14 \quad q = 63 \quad r = -90$

$\therefore (u_1 + u_2) = 3pq - 9r = -1836$

$u_1 u_2 = 90^3 + 9\left\{\left(\frac{p}{3} - \frac{q}{4}\right)r + 11r^2\right\}$

$= 2250423 - 206388 + 65610 = 874263$

$u - u_2 = -18$

$\therefore u_2 = -909 \quad u_1 = -927 \quad u_0 = -908$

Then $x_1 = -908 - \frac{927}{2}\{-1 + \sqrt{-3}\} - \frac{909}{2}\{-1 - \sqrt{-3}\}$

$= 19 - 9\sqrt{-3}$

Similarly $x_2 = 19 + 9\sqrt{-3}$

$\therefore$

$\therefore \sqrt[3]{u_1} = \sqrt[3]{10}\sqrt[3]{1 - \frac{7}{10}\sqrt{-3}} = \sqrt[3]{10}\left\{1 - \frac{3}{10}\sqrt{-3} + \frac{\frac{1}{3}\cdot\frac{-2}{3}}{1\cdot 2}\cdot\frac{9\cdot 9\cdot -3}{10\cdot 10\cdot 10} - \frac{\frac{1}{3}\cdot\frac{-2}{3}\cdot\frac{-5}{3}}{1\cdot 2\cdot 3}\cdot\frac{9\cdot 9\cdot 9\cdot -3\sqrt{-3}}{10\cdot 10\cdot 10\cdot 10} + \frac{\frac{1}{3}\cdot\frac{-2}{3}\cdot\frac{-5}{3}\cdot\frac{-8}{3}}{1\cdot 2\cdot 3\cdot 4}\left(\frac{9}{10}\right)^4 \cdot -3 \right.$

$+$

$\sqrt[3]{10}\left\{1 + .27 - .243 + .33679 - .5501 \ldots - \frac{7}{10}\sqrt{-3}\right\}$

where lastly the first three only are —

$x_1 = 2.2\{1.063 - .4\sqrt{-3}\} = 2.3386 - .4\sqrt{-3}$

and $x_2 = \qquad\qquad 2.3386 + .4\sqrt{-3}$

$x = \frac{1}{3}\{-14 + 4.777\} = -\frac{7}{3} = -3$, nearly

which gives one after root a decimal nearer the true root.

Cubic Equations.

$(x-4)(x-3)(x-7) = x^3 - 14x^2 + 61x - 84.$

Here assume $u_1 + u_2 = 3k_2 - 9r = 2562 - 756 = 1806$ (i)

$u_1 u_2 = 9q^3 + 9\{k^3 - 6kq\} + 8r^2$

$= 204282 - 179280 + 57536 = 815085$ (ii)

Then by squaring (i) and subtracting 4 times (ii), we have

$u_1 - u_2 = 36$

$\therefore u_1 = 903 + 18 = 921, \quad u_2 = 885$

also $u_3 = k_3 - (u_1 + u_2) = 2744 - 1806 = 938$

Then $z_1 = 938 + \frac{921}{2}\{-1 + \sqrt{-3}\} + \frac{885}{2}\{-1 - \sqrt{-3}\}$

$= 938 - 903 + 18\sqrt{-3} = 35 + 18\sqrt{-3}$

Similarly $z_2 = \quad\quad\quad\quad = 35 - 18\sqrt{-3}$

$\therefore \sqrt[3]{z_1} = \sqrt[3]{35 + 18\sqrt{-3}} = \sqrt[3]{35}\sqrt[3]{1 + \frac{18}{35}\sqrt{-3}}$

$= 3.22\left\{1 + \frac{6}{35}\sqrt{-3} + \frac{1}{3}\cdot\frac{324}{1225} + \&c. \right\}$

$= 3.22 \times 1.08 - + \frac{6}{35}\sqrt{-3}$

$= 3.4776 + \frac{6}{35}\sqrt{-3}$

Similarly $\sqrt[3]{z_2} = 3.4776 - \frac{6}{35}\sqrt{-3}$

Therefore denoting by a one of the roots of the equation we have

$a = \frac{1}{3}\{k + \sqrt[3]{z_1} + \sqrt[3]{z_2}$

$= \frac{1}{3}\{14 + 6.9552\} = \frac{20.9552}{3} = 7$ nearly, which is a root

and dividing the proposed equation by $x - 7$ it will be depressed to a quadratic ——

$(x+b)^n = a^n + n a^{n-1} b + \frac{n(n-1)}{1.2} a^{n-2} b^2 + \frac{n(n-1)n-2)}{1.2.3} a^{n-3} b^3 + \&c$

of which the coeffs are

$$1 + n + \frac{n(n-1)}{1.2} + \frac{n(n-1)n-2)}{1.2.3} + \&c$$

Therefore $x = \eta$. $\beta = n(n-1)$ ——

and by substitution in the series

$1 + \frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{4}$ —— we have

$1 + \frac{n}{2} + \frac{n(n-1)}{1.2.3} + \frac{n(n-1)(n-2)}{1.2.3.4} + \&c = S$ suppose

i. $n+1 + \frac{(n+1)n}{2} + \frac{(n+1)n(n-1)}{1.2.3} + \frac{(n+1)n(n-1)(n-2)}{1.2.3.4} \&c = S(n+1)$

$1 + n+1 + \frac{(n+1)n}{2} + \&c$ —— $= S(n+1) + 1$

or $(1+1)^{n+1} = S(n+1) + 1$

$2^{n+1} - 1 = S(n+1)$

$\frac{(2^{n+1} - 1)}{(n+1)} = S$

$$\frac{mx+a}{ny} = \frac{(m+n)y}{(m+n)x+a} \quad \therefore \quad \frac{m(x+y)+a}{m(x-y)+a} =$$

Volles Quadrat

$$\frac{1+\frac{a}{m}}{y} = \frac{y}{x+\frac{a}{m+n}} \quad : \quad y^2 = \left(x+\frac{a}{m}\right)\left(x+\frac{a}{m+n}\right)$$

$$(m+n)^2 y^2 = b^2+a^2+2ax(m+n)+(m+n)^2 x^2$$

$$y^2 = \frac{b^2+a^2+2ax(m+n)+(m+n)^2 x^2}{(m+n)^2} = x^2 + \frac{2a}{m+n}\cdot x + \frac{b^2+a^2}{(m+n)^2}$$

$$x^2 + x\left(\frac{a}{m+n}+\frac{a}{m}\right) + \frac{a^2}{n(m+n)} = x^2 + \frac{2a}{m+n}x + \frac{b^2+a^2}{(m+n)^2}$$

$$x\,\frac{am+a(m+n)}{m(m+n)} = \left(\frac{2a}{m+n}\right)x + \frac{b^2+a^2}{(m+n)^2}$$

$$x\left(\frac{2am+an}{m}-2a\right) = \frac{b^2+a^2}{m+n}$$

$$x\cdot\frac{2am-an}{m} =$$

$$x = \frac{b^2+a^2}{m+n} \times \frac{m}{(2am-an)} = \frac{mb^2+ma^2}{a(2m^2+mn-n^2)}$$

$$x\left(\frac{2am+an}{m}-2a\right) = \frac{b^2+a^2}{m+n} \qquad a(2m-n)$$

$$a(2m^2+mn-2m-n^2)$$

$$a(2m^2+mn-n^2)$$

$$x\left(\frac{an}{m}\right) = \frac{b^2+a^2}{m+n}$$

$$x = \frac{m(b^2+a^2)}{an(m+n)}$$

$$y^2 = \left(\frac{m(b^2+a^2)}{an(m+n)}+\frac{a}{m}\right)\left(\frac{m(b^2+a^2)}{an(m+n)}+\frac{a}{m+n}\right.$$

A shepherd counted his flock by 2. 3. 4. 5. 6. & a bo
and there was 1 over, but when he counted them by a bun
there were none over. Reqd. the least no. of sheep in the flock

Let x = numb $\therefore \frac{x-1}{2}$ = a whole no. or $x = 2p+1$

also $\frac{x-1}{3} = \frac{2p}{3} = q$ a whole number, or $2p = 3q$ or $p = q + \frac{q}{2}$

Let $\frac{q}{2} = r$ Then $q = 2r$, $p = 3r$ and $x = 6r+1$

also $\frac{x-1}{4} = \frac{6r}{4}$ a whole no. or $r + \frac{r}{2}$ a whole no. $\therefore \frac{r}{2}$ a whole no.

Let $\frac{r}{2} = m \therefore r = 2m$ and $x = 12m+1$

also $\frac{x-1}{5} = \frac{12m}{5}$ a whole no, or $2m + \frac{2m}{5}$ = a whole no. $\therefore \frac{2m}{5}$ = a whole no

Let $\frac{2m}{5} = 2n \therefore m = 5n$ and $x = 60n+1$

also $\frac{x-1}{5} = \frac{60n}{5} = 12n$ which is evidently a whole no.

also $\frac{x}{7} = \frac{60n+1}{7}$ a whole no. $\therefore 8n + \frac{4n+1}{7}$ = a whole no. $\therefore \frac{4n+1}{7}$ = a whole no.

Let $\frac{4n+1}{7} = s \therefore 4n+1 = 7s$ or $n + \frac{1}{4} = s + \frac{3s}{4} \therefore n = s + \frac{3s-1}{4}$

Let $\frac{3s-1}{4} = u \therefore 3s-1 = 4u$ or $s - \frac{1}{3} = u + \frac{u}{3} \therefore s = u + \frac{u+1}{3}$

Let $\frac{u+1}{3} = v \therefore u+1 = 3v$ or $u = 3v-1$

also $s = u + \frac{u+1}{3}$ or $s = 4v-1$

and $n = s + \frac{3s-1}{4}$ or $n = 7v-2$

But $x = 60n+1$ — or $x = 420v - 120 + 1$

Now let $v = 1$

Then $x = 301$

Let $v = 2$

Then $x = 741$ &c

$\therefore$ the least value of $x = 321$

$$x^2 - yz + z^2 : \frac{y^3 + z^3}{y+z} : \frac{5}{x}$$

$$x^2 + 2yz + z^2 = x^2$$

$$3yz : x = \frac{5}{x}$$

Indeterminate Equations.

1, $+x+y=79$ $\therefore xy+y=79-x$ or $y=\frac{79}{x+1}-\frac{x}{x+1}=-1+\frac{80}{x+1}$

from which it is manifest that $x+1$ must be a divisor of 80.

Such 80 having several divisors we shall have several values of x

Divisors – 1. 2. 4. 5. 8. 10. 16. 20. 40. 80. and $x=0.1.3.4.7.9.15.19.39.79$

$y=79.39.19.15.9.7.4.3.1.0$

By the answers in the bottom line we see the same value in x & y fits in with any x & y in the top line. $x=$ [illegible] $y=$ [illegible]

Similarly if the eq.^{n} be $xy+ax+by=c$ we have $(x+b)y=c-ax$ or $y=\frac{ab+c}{x+b}-a$

$x+b$ must be a divisor of the known quantity $ab+c$, & the several divisors of the number give a value of x. Assume $ab+c=f$ $\therefore y=\frac{f}{x+b}-a$

$x+b=f$ or $x=f-b$ then $y=g-a$: consequently there are two answers for respectively the number $ab+c$ by a process of division of f & g.

Ex. $5xy=2x+3y+15$ or $y=\frac{2x+15}{5x-3}$ $\therefore y=\frac{10x+30}{5x-3}=2+\frac{36}{5x-3}$

It is therefore required that those divisors of 36 which added to 3 will give numbers divisible by 5.

Only 2. 12. 32 will answer the condition

if $5x-3=2$, $5y=50$ or $x=1$, $y=10$

if $5x-3=12$, $5y=10$ or $x=3$ $y=2$

if $5x-3=32$, $5y=5$ or $x=7$, $y=1$

Ex. $3xy-6y=-3x+11$

$(3x-6)y=-3x+11$ or $y=-1+\frac{10}{3x-6}$

x 10 is divisible by $3x-6$ – it is required therefore to find those divisors of 10 which increased by 6 are divisible by 3.

2 and 5 answer the condition, and the corresponding values of x are 2. 3

if $y=$ — 4. 2.

Ex. $xy+x^2=2x+3y+29$

$y=\frac{2x-x^2+29}{x-3}=-x-1+\frac{26}{x-3}$

$\therefore y+x+1=\frac{26}{x-3}$ or $x-3$ is a divisor of 26

The divisors of 26 are 1. 2. 13. 26 and we have the following answers

$x-3=1$, or $x=4$, $y=21$

$x-3=13$, or $x=16$, $y=-\frac{15}{16}$

$x-3=2$, or $x=5$, $y=\frac{1}{2}$

The 2nd value $\frac{15}{16}$, being negative must be rejected, since for the same reason we cannot include the last case $x-13=-1$.

The Indeterminate Equation

Indeterminate Equations

$5x^2 - 5y = 10x + 5$, $y = \frac{x+1-2x^2}{x-1-2x}$ $2xy = \frac{2x^2-x-x^2}{x-1-2x} = x + 1 + \frac{1}{x+1}$

common divisor is 1 and x is positive integer only when x is equal to 1 or 2

$\therefore 2x = -1$ $\therefore 2xy = -1+1+2 = 2$ $\therefore xy = 1$

$\therefore 2x = -1$ $\therefore 2xy = -6+3+2 = -1$ or $y = -3$

and these are the only integral values of x and y.

The nth term of the series $= n\{[(-1)^n+1]n+2\}$ and as this is true for every value of n the pth term would be...

$$p\{[(-1)^p+1]p+2\} = 2p^2+2p \text{ or } 2p \text{ according as } p \text{ is even or odd} —$$

& hence the $2p^{th}$ term $= 2p\{[(-1)^{2p}+1)2p+2\} = 8p^2+4p$ —

Taking p therefore as an even no:

the pth term $= 2p^2+2p$, and the $2p^{th} = 8p^2+4p$

and the diff between these gives p times the common diff.

$$\text{or } d = \frac{8p^2+4p-(2p^2+2p)}{p} = \frac{6p^2+2p}{p} = 6p+2 —$$

To find a

$$a+(p-1)d = p^{th} \text{ term}$$

$$a+(p-1)6p+2 = 2p^2+2p$$

$$a = 2p^2+2p-6p^2+6p-2p+2$$

$$= 6p-4p+2$$

$$\text{also } S = \left(\frac{a+l}{2}\right)n$$

$$= \left(\frac{6p-4p^2+2+8p^2+4p}{2}\right)2p$$

$$= (4p^2+10p+2)p$$

---

2. g. The sum of $2p$ terms of a series whose n^{th} term

$$n\{[(-1)^n+1)n+2\} —$$

$xy = a(x+y)$

$xz = b(x+z)$

$yz = c(y+z)$

Let $x = vy$

$\therefore vy^2 = a(v+1)y$ $\quad v(y^2 - ay) = ay = \frac{b}{c}\cdot vy + z$

$vyz = b(vy+z)$

$yz = c(y+z)$

$v = \frac{b}{c}\left(\frac{vy+z}{y+z}\right)$

$cv(y+z) = b(vy+z)$

$v(cy - by) = bz$

$v(cy + cz - by) = bz$

$v = \frac{bz}{cy+cz-by} = \frac{ay}{y^2-ay}$

$bz(y^2 - ay) = (cy + cz - by)ay$

$y^2(bz - ca + ba) = (bza + cza)y$

$y(z-c) = zc$

$y = \frac{zc}{z-c}$ (II) $= \frac{(b+c)za}{bz - ca + ba}$ $\qquad y = \frac{(b+c)za}{bz-ca+ba}$ (I)

$\frac{c}{z-c} = \frac{(b+c)a}{bz + (b-c)a}$

$cbz + (b-c)ac = (b+c)az - (b+c)ac$

$(cb - ab - ac)z = -2ac(b) + (c-b-b-c)ac = -2b\cdot ac$

$(ac + ab - cb)z = 2acb$

$z = \frac{2abc}{ac+ab-cb}$

$y = \frac{zc}{z-c} = \frac{2abc}{ab+bc-ac}$

$x = \frac{xc}{z-b} = \frac{2abc}{bc+ac-ab}$

The answer might have been reached by getting a value of z from the first two equations and equating it with that found in the last ——

$$\frac{1^2}{1.3.4.6} + \frac{2^2}{3.4.5.7} + \frac{n^2}{(n+1)(n+2)(n+3)(n+4)}$$

$$\Delta S = \frac{(n+1)^2}{(n+2)(n+3)(n+4)(n+6)} = \frac{(n^2+2n+1)(n+5)}{(n+2)(n+3)(n+4)(n+5)(n+6)}$$

$$= \frac{(n+1)(n+5)}{(n+1)(n+2) \cdots (n+6)}$$

$$= \frac{(n+2)(n+3)(n+4) - 2(n+2)(n+3) - 5(n+2) + 9}{(n+2)(n+3)(n+4)(n+5)(n+6)}$$

$$= \frac{1}{(n+5)(n+6)} - \frac{2}{(n+4)(n+5)(n+6)} - \frac{5}{(n+3)(n+4)(n+5)(n+6)}$$

$$+ \frac{9}{(n+2)(n+3)(n+4)(n+5)(n+6)}$$

$$S = -\frac{1}{n+5} + \frac{2}{2.(n+4)(n+5)} + \frac{5}{3} \frac{1}{(n+3)(n+4)(n+5)}$$

$$- \frac{9}{4} \frac{1}{(n+2)(n+3)(n+4)(n+5)} + C$$

$$S = -\frac{1}{5} + \frac{1}{20} + \frac{5}{180} - \frac{9}{480} + C$$

$$= -\frac{1}{5} + \frac{1}{20} + \frac{1}{36} - \frac{1}{160} + C$$

$$\frac{157}{1440} =$$

$$S = -\frac{1}{n+5} + \frac{1}{(n+4)(n+5)} + \frac{5}{3} \frac{1}{(n+3)(n+4)(n+5)} + \frac{9}{4} \frac{1}{(n+2)(n+3)(n+4)(n+5)}$$

$$+ \frac{157}{1440}$$

$$\therefore \sigma = \frac{157}{1440}$$

Summation of series

$$\mathfrak{L}_2^n \cdot \mathfrak{L}_3^n \; \mathfrak{L}_4^n \cdots \langle \mathfrak{L}_{n-}^n \rangle = \langle \mathfrak{L}_3 \; \mathfrak{L}_{n-1} \left\{ \left(\mathfrak{L}_3^n - \mathfrak{L}_2^n\right)\left(\mathfrak{L}_4^n - \mathfrak{L}_3^n\right)\left(\mathfrak{L}_5^n - \mathfrak{L}_4^n\right) \cdots \cdots \left(\mathfrak{L}_{n-1}^n - \mathfrak{L}_{n-2}^n\right)\right.$$

$$\mathfrak{L}_2^n \cdot \mathfrak{L}_4^n \; \mathfrak{L}_5^n \cdots \underset{n-2}{\mathfrak{L}^n} \cdot \underset{n}{\mathfrak{L}^n} \rangle = \langle \left(\mathfrak{L}_3^n - \mathfrak{L}_2^n\right)\left(\mathfrak{L}_4^n - \mathfrak{L}_3^n\right) \cdots \left(\underset{n-2}{\mathfrak{L}^n} - \underset{n-3}{\mathfrak{L}^n}\right)\left(\underset{n-1}{\mathfrak{L}^n} - \underset{n-2}{\mathfrak{L}^n}\right)$$

$$\mathfrak{L}_2^n \cdot \mathfrak{L}_4^n \; \mathfrak{L}_1^n \cdots \underset{n-2}{\mathfrak{L}^n} \cdot 1.2.3 - n \rangle = \langle \left(\mathfrak{L}_3^n - \mathfrak{L}_2^n\right)\left(\mathfrak{L}_4^n - \mathfrak{L}_3^n\right) \cdots \left(\underset{n-2}{\mathfrak{L}^n} - \underset{n-3}{\mathfrak{L}^n}\right)\left(2.3.4 \cdots n - 3.4 \cdots n\right)$$

$$n(n-1)\,\mathfrak{L}_4^n \cdot \mathfrak{L}_5^n \cdots \underset{n-2}{\mathfrak{L}^n} \cdot 1.2.3 \cdots n \rangle = \langle \left\{n(n-1)\,n-2\right\} - \left\{n(n-1)\right\}\left(\mathfrak{L}_4^n - \mathfrak{L}_3^n\right)\left(-\underset{n-2}{\mathfrak{L}^n} - \underset{n-3}{\mathfrak{L}^n}\right) 3.4. \cdots n$$

$$n(n-1)\left\{\left\{n(n-1)(n-2)\,n-3\right\}\mathfrak{L}_5^n - \underset{n-2}{\mathfrak{L}^n} \times 2\right\} = \langle \left\{n(n-1)(n-2)\left\{\left\{n(n-1)(n-2)(n-3)\right\} - \left\{n(n-1)(n-2)\right\}\right\}\mathfrak{L}_5^n - \mathfrak{L}_4^n\right\} - \left(\underset{n-2}{\mathfrak{L}^n} - \underset{n-3}{\mathfrak{L}^n}\right)$$

$$\left\{n(n-1)(n-2)(n-3)\right\} n(n-1)(n-2)(n-3)(n-4)\left\{\mathfrak{L}_6^n - \underset{n-2}{\mathfrak{L}^n}\right\} 2 \rangle = \langle (n-3)\left\{n(n-1)(n-2)(n-4)\right\}\left\{n(n-1)(n-2)(n-3)(n-5) \cdots \cdots \left(3.4 \cdots n - 4.5 \cdots n\right)\right.$$

$$n(n-1)(n-2)(n-3)(n-4)\left\{\mathfrak{L}^{n-5} \cdots \cdots 3.4 \cdots n \times 2\right\} = \langle \; n-4\,(n(n-1)(n-2)(n-3)(n-5) \cdots \cdots \mathfrak{L}^{n-4.n-6} \cdot \left(2.4.5 \cdots n\right)$$

$$\mathfrak{L}^{n-5} \cdot \mathfrak{L}^{n-6} \cdots 4.5 \cdots n \times 3 \rangle = \langle \left(n-5\right)\left(\mathfrak{L}^{n-4.n-6}\right)\left(\mathfrak{L}^{n-5}(n-7)\right) \cdots \left(3.5.6 \cdots n\right)$$

And since n is a finite quantity the quantities $\mathfrak{L}^{n-5}$. $\mathfrak{L}^{n-6}$ at one end of the series of the terms 3.4 — n, 4.5 — n at the other end must continually approach nearer to each other, until they meet. The same remark is also applicable to the contrary extreme of the series on the other side of the eq? — so that at last we shall have.

$$\mathfrak{L}^{n-r}\; \mathfrak{L}^{n-(r+1)} \cdots \overset{\mathfrak{L}^{n-(r+2)}}{\wedge} \rangle = \langle \mathfrak{L}^{n-r} \cdot \mathfrak{L}^{n-(r+1)} \cdot \left\{n-(r+2)\right\}\left(n-(r+3) \times n-(r+1)\right) n-r \cdots n.$$

$$\mathfrak{L}^{n-(r+2)} \rangle = \langle \left(n-(r+2)\right)\left(n-r+3 \times n-(r+1)\right)(n-r) \cdots n$$

$$n(n-1) \cdots (n-2) \cdots n-(r+3) \rangle = \langle \; n(n-1) \cdots \left(n-r\right)\left\{n-(r+1)\right\}\left\{n-(r+2)\right\}\left\{n-(r+3)\right\}$$

By the latter terms are identical consequently also

$$\mathfrak{L}_2^n \cdot \mathfrak{L}_3^n \cdots \underset{n}{\mathfrak{L}^n} = \mathfrak{L}_3 \; \underset{n-1}{\mathfrak{L}^n}\left\{\left(\mathfrak{L}_3^n - \mathfrak{L}_2^n\right)\left(\mathfrak{L}_4^n - \mathfrak{L}_3^n\right) \cdots \cdots \left(\underset{n-1}{\mathfrak{L}^n} - \underset{n-2}{\mathfrak{L}^n}\right)\right. \text{ which was to be proved}$$

The solution of two eq^{ns} of the second degree between two unknown quantities, depends on the solution of an eq^{n} of the 4th degree with respect to one unknown quantity of a degree not greater than 4 as before.

For in the general eq^{n} of the second degree

$$\left.\begin{aligned} ay^2 + bxy + cx^2 + dy + ex + f &= 0 \\ a'y^2 + b'xy + c'x^2 + d'y + e'x + f' &= 0 \end{aligned}\right\}$$ by eliminating x^2 and equating the results we have

$$c'(ay^2 + bxy + dy + ex + f) = c(a'y + b'xy + d'y + e'x + f')$$

$$\therefore x = \frac{(ca' - ac')y^2 + (c'd - d'c)y + c'f - fc'}{(bc' - cb')y + ec' - ce'}$$

since therefore the numerator of this expression is of the second degree in y, its square is of the fourth degree. and when the expression is substituted for x in either of the given eq^{ns} the resulting eq^{n} is of the 4th degree

$x - 2\sqrt{x+2} = 1 + \sqrt{x^3 - 3x + 2}$

8.2.13

$x - 1 - 2\sqrt{x+2} = \sqrt{x^3 - 3x + 2}$

$(x-1)^2 + 4(x+2) - 4(x-1)\sqrt{x+2} = \sqrt{x^3 - 3x + 2} = (x-1)\sqrt{x+2}$

$(x-1)^2 + 4(x+2) = 5(x-1)\sqrt{x+2}$

$(x-1)^2 - 5\sqrt{x+2}(x-1) = -4(x+2)$

$(x-1)^2 - 5\sqrt{x+2}(x-1) + \frac{25}{4}(x+2) = \frac{9}{4}(x+2)$

$x - 1 - \frac{5}{2}\sqrt{x+2} = \frac{3}{2}\sqrt{x+2}$

$x - 1 = 4\sqrt{x+2}$

$x^2 - 2x + 1 = 16x + 32$

$x^2 - 18x = 31$

$x^2 - 18x + 81 = 31 + 81 = 112$

$x - 9 = \sqrt{112}$

$x = 9 + \sqrt{112}$

9/2

Let S & H be the hour $\angle$s n hours before noon then $\angle\ HPn = 15n$

and $SPH = Q - 15n$

In rt $\angle^d\ \triangle\ PSH$

$$\tan SP = \cos SH \cot SPH$$

or $$\cos SH = \tan SP \cot(Q - 15n)$$

Whence given n the nat. $\angle$s, 2, 3 — the hour $\angle$s may be drawn

Cubic Equations

$(x-6)(x-2)(x-20) = x^3 - 28x^2 + 172x - 240$ —

Here $u_1 + u_2 = 3pq - 9r = 1464 - 2160 = 12288$,

$u_1 u_2 = 9q^3 + 9(p^3 - 6pq)r + 81r^2$

$= 4579632 - 14999040 + 4665600$

$= 3546192$.

$u_1^2 + 2u_1u_2 + u_2^2 = 150994944$

$-4u_1u_2 = -141850368$

$u_1^2 - 2u_1u_2 + u_2^2 = 9144576$

$u_1 - u_2 = 3024 \quad \therefore u_1 = 7656,\ u_2 = 4632$

$u_0 = p_3 - (u_1 + u_2) = 21952 - 12288 = 9664$

Now $z_1 = u_0 + \alpha u_1 + \alpha^2 u_2$

$= 9664 + 7656\left(-\frac{1}{2} + \frac{\sqrt{-3}}{2}\right) + 4632\left(-\frac{1}{2} - \frac{\sqrt{-3}}{2}\right)$

$= 9664 - 6144 + 1512\sqrt{-3}$

$= 3520 + 1512\sqrt{-3}$

and $z_2 = 3520 - 1512\sqrt{-3}$

$\sqrt[3]{z_1} = \sqrt[3]{3520 + 1512\sqrt{-3}} = \sqrt[3]{3520}\sqrt[3]{1 + \frac{189}{440}\sqrt{-3}}$

$= 15.2\left\{1 + \frac{1}{3}\cdot\frac{189}{440}\sqrt{-3} + \frac{1}{9}\cdot\frac{189\times189\times3}{440\cdot440}\right.$

$= 15.2 \times 1.0567 + \frac{63}{440}\sqrt{-3}$

$= 16 \text{ nearly} + \frac{63}{440}\sqrt{-3}$

similarly $\sqrt[3]{z_2} = 16 \text{ nearly} - \frac{63}{440}\sqrt{-3}$

and if α, β, γ are the 3 roots of the eqn. we have

$\alpha = \frac{1}{3}\left\{p + \sqrt[3]{z_1} + \sqrt[3]{z_2}\right) = \frac{1}{3}\left\{28 + 16 + \frac{63}{440}\sqrt{-3} + 16 - \frac{63}{440}\sqrt{-3}\right\}$

and if the given eqn. be divided by $x-20$ it will be reduced to a quadratic —

$= \frac{1}{3}\times 60 = 20.$

$(x-2)(x-3)(x-4) = x^3 - 9x^2 + 26x - 24$

Here $u_1 + u_2 = 3fg - 9r = 486$

$u_1 u_2 = 9g^3 + 9(h^3 - 6fg)r + 81r^2 = 59040$

$u_1 = 246$, $u_2 = 240$ $u_0 = 243$.

$z_1 = u_0 + u_1 a + u_2 a^2$

Here as $a = -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ and $a^2 = -\frac{1}{2} - \frac{1}{2}\sqrt{-3}$ two roots of unity.

$z_1 = 243 + \frac{246}{2}(-1+\sqrt{-3}) + \frac{240}{2}(-1-\sqrt{-3})$

$= 243 - 243 + 3\sqrt{-3}$ $\therefore z_1 = 3\sqrt{-3}$

$z_2 = 243 + \frac{240}{2}(-1+\sqrt{-3}) + \frac{246}{2}(-1-\sqrt{-3})$

$= 243 - 243 - 3\sqrt{-3}$ $\therefore z_2 = -3\sqrt{-3}$

and we have $\sqrt[3]{z_1} = \sqrt[3]{3\sqrt{-3}}$, $\sqrt[3]{z_2} = \sqrt[3]{-3\sqrt{-3}}$

and if a.b.c be the roots of the eq^n. we have

$a = \frac{1}{3}\{f + \sqrt[3]{z_1} + \sqrt[3]{z_2}\}$ &c $\frac{3}{3}$.

$= \frac{1}{3}\{g + \sqrt[3]{3\sqrt{-3}} + \sqrt[3]{-3\sqrt{-3}}\} = \frac{9}{3} = 3$

and dividing the equation by $x-3$ we reduce it to a quadratic ——

Algebra.

Prove the rule for the multiplication of numbers by casting out the nines, and shew that the product of any n consecutive numbers is divisible by $1.2.3 \ldots n$. Also find a number which is equal to the sum of all the numbers which divide it without a remainder.

Find the integral values of x and y which satisfy an equation of the form $ax + by = c$. Ex. $5x + 7y = 29$.

Sum the following series

$$\frac{n-1}{n}x + \frac{n-2}{n}x^2 + \frac{n-3}{n}x^3 \ldots - 1^2 + 2^2 + 3^2 \ldots - 1^3 + 2^3 + 3^3 \ldots \text{to } n \text{ terms}$$

and shew that $n^n - n(n-1)^n + \frac{n(n-1)}{2}(n-2)^n \ldots = 1.2.3 \ldots n$.

Solve the equation $ax + \sqrt{bx + c} = d$ and explain the result.

When a cube root consists of $2n+2$ digits, and $n+2$ have been found by the ordinary rule the rest can be found by division.

$$D = P\left(1 + \frac{I}{R^n}\right) \quad ; \quad D = \frac{PI}{P + I}.$$

Prove the general properties of quadratic eqns. $\S$ 76. Art. 44.

Reduce the minimum of $\frac{a^2x^2 + b^2}{(a^2 + b^2)x}$ when $a \neq b$

Prove the rules for Single and Double position.

If a be the number whose cube is the nearest to a given number then $\sqrt[3]{m} = a\left\{\frac{2m + a^3}{m + 2a^3}\right\}$ Apply this to the cube root of 28.

Find a quantity which multiplied by $\sqrt{a} + \sqrt{b} + \sqrt{c}$ will produce a rational result.

If $\sqrt{x} + \sqrt{y} = a + \sqrt{b}$ then $x = a$ and $y = b$

Find the coeff^{t} of x^5 in $(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5)^2$

and of x^6 in $(1 + x + 2x^2 + 3x^3 + 4x^4 + 6x^5)^2$

From a vessel containing b measures of wine a measures are taken and c measures of water are added — find what quantity of wine will remain after the process has been repeated 3 times.

St. Johns College.

Reduce 8″.57116 to the decimal of a degree; and find the value of $36^{d}.24.22218$.

If 3 miles 4 furlongs and 93 yards be run in $6^{m}.4^{s}$ how much is that short of the rate of a mile per minute?

Reduce to lowest terms $\dfrac{a^3 - 3a^2b + 3ab^2 - b^3 - ac^2 + bc^2}{2a^2b + 2ac^2 + 2bc^2 - a^4 - b^4 - c^4}$

Simplify the expressions

$$\sqrt[3]{8+16\sqrt{5}} - \sqrt[3]{1+\sqrt{20}}, \quad (\sqrt{5}-1+\sqrt{-10-2\sqrt{5}})^5$$

Two persons walking round a triangular field in opposite directions observe which are to one another as 13 to 11, and starting together from the right angle meet first in the middle point of the opposite side and afterwards 20 yards from the point where they started — required the sides of the field.

Find the depth of a well for sinking which £103.11.0 was paid being at the rate of 9s for the first foot and an additional 10d for every succeeding foot — explain the negative answer

Expand $(a^2 - x^2)^{\frac{1}{2}}$ as far as four terms by the Binomial Theorem, and deduce the coeff. of x^m. Also find when the series for $(1+\frac{7}{10})^{-5}$ begins to converge.

Find the sum of n terms of a series whose n^{th} term is

$$\left\{ na + \frac{n(n-1)}{2} b \right\} r^{n-1}$$

Shew that if $x = \dfrac{m^2 - cn^2}{bn^2 - 2amn}$, then $ax^2 + bx + c$ is a perfect square; also investigate a value of x which makes $ax^2 + c$ a perfect square.

If $x = 2na + b$ or $= (2n+1)a - b$ where n is any integer shew how all the values of x may be comprised under one formula.

Shew that the value of $1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3}$ &c. infinite differs from the sum of the first $(n+1)$ terms by a quantity less than $\frac{1}{n^2 \lfloor n-1}$; and that the error in taking only 11 terms does not affect the seventh decimal place.

Having given the number of figures in any number of factors to find the number of figures in their continued product.

Suppose n factors a b c &c each of which α contains
α figures a is less than 10^{α} but not less than $10^{\alpha-1}$
β contains β figures ∴ b < 10^{β} but not less than $10^{\beta-1}$
then let α+β+γ+&c = n

$10^{n} = x$ is least value of a×b×c×&c
10^{n} > than greatest value of the product

∴ a×b×c×&c has fewer than n+1 figures but not fewer than n−x+1
Hence if a = b = c = &c α = β = γ = &c n = nα
we find that if a contain α figures a^{x} will have at least
$x^{\alpha}(\alpha-1)+1$ but not more than $x^{\alpha}\alpha$
Again if there be x figures in $\sum a$ then α is not less than
$\alpha(x-1)+1$ and not greater than αx therefore x is not less
than $\frac{\alpha}{n}$ and not greater than $\frac{\alpha+n-1}{n}$

Ex: A number of 21 figures will have not less than 101 nor
more than 105 figures in its 5th power & of 85 figures in its 5th root

---

When 2 numbers a b are given approximately ~~exact~~ and limits
of errors are known show how to find limits of errors in their
product

Suppose p and q approximate value of a and b so that
a = p+x b = q+y pq+z
then ab = pq + py + qx + xy = then if e be the greatest value
which x and y can have e(p+q+e) will be the greatest
value z can have thus the limit of error may be easily found
Suppose p = 10.31 q = 1.29 e = .004 then the greatest value
error in their product = e(p+q+e) = 11.604 × .004 = .046416
If the error be in defect e must be used

Prove that every vulgar fraction of the form $\frac{a}{2^h 5^g}$ where h g are nothing or any positive integer may be reduced to a terminating decimal and also what is the number of decimal places

$$\frac{a}{2^g 2^h 5^g} = \frac{a\,5^{h-g}}{2^h 5^g 5^{h-g}} = \frac{a\,5^{h-g}}{2^h (5^{h+g-g})} = \frac{a\,5^{h-g}}{10^h}$$ which is a terminating decimal having h decimal places. (number of)

When a vulgar frac. cannot be reduced to a terminating decimal the figures in the quotient cannot be greater than $b-1$ (and are b in number)

If a and b are the sum and difference of 2 quantities prove that $a^2 - b^2$ is divisible by 4

Let $a = x+y$, $b = x-y$ then $a^2 - b^2 = (x+y)^2 - (x-y)^2 = 4xy$ which is divisible by 4 $\therefore$ Q.E.D.

Add following quantities and show that all values are positive, x y and z.

$$\left.\begin{array}{l} x^2 - \frac{1}{9}y^2 + \frac{1}{4}z^2 + bz \\ \frac{4}{9}y^2 - \frac{1}{2}x^2 - \frac{2}{3}by + a^2 \\ ax - \frac{1}{2}bz + \frac{7}{12}b^2 - \frac{1}{2}a^2 \end{array}\right\} = \frac{1}{2}x^2 + ax + \frac{1}{2}a^2 + \frac{1}{3}y^2 - \frac{2}{3}by + \frac{1}{3}b^2 + \frac{1}{4}z^2 + \frac{1}{2}bz$$

$$= \frac{x^2 + 2ax + a^2}{2} + \frac{y^2 + 2by + b^2}{3} + \frac{z^2 + 2bz + }{4}$$

$$= \frac{(x+a)^2}{2} + \frac{(y-b)^2}{3} + \frac{(z+b)^2}{4}$$

Multiply $\overline{ax+b^2} \times \overline{cx+d^2}$ find what relation must exist between a b c and d in order that the product may be a perfect square for all values of x

$\overline{ax+b^2} \times \overline{cx+d^2} = acx^2 + ad^2x + b^2cx + b^2d^2$ and in order that this may be a perfect square $4acb^2d^2 = (ad^2 + b^2c)^2 = a^2d^4 + b^4c^2 + 2a$ independent of x and by solving this equation it becomes $ad^2(b^2c - ad^2) = b^2c(b^2c - ad^2)$

$$\therefore ad^2 = b^2c$$

$$\text{or } \frac{a}{c} = \frac{b^2}{d^2}$$

$\frac{n}{d} = 1 + 2x + 3x^2 + 4x^3 + \&c \quad + nx^{n-1}$

The general relation between 3 consec: coeff $R.\ S.\ T$ is $T - 2R + R = 0$

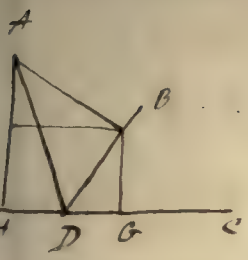
$\therefore d = 1 - 2x + x^2$

$\therefore n = (1 - 2x + x^2)(1 + 2x + 3x^2 + \&c$ to n terms$)$

$= 1 - (n+1)x^n + nx^{n+1}$

$\therefore \frac{n}{d} = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2}$

When the series is infinite n vanishes as also the terms which involve it



Then more directly Prob. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$A+B$ being greater than 90°

Let $BDC = A$, $ADB = B$ $\therefore ADC = (A+B)$

Take any point A in AD. draw $AB \perp$ to DB & complete the construct.

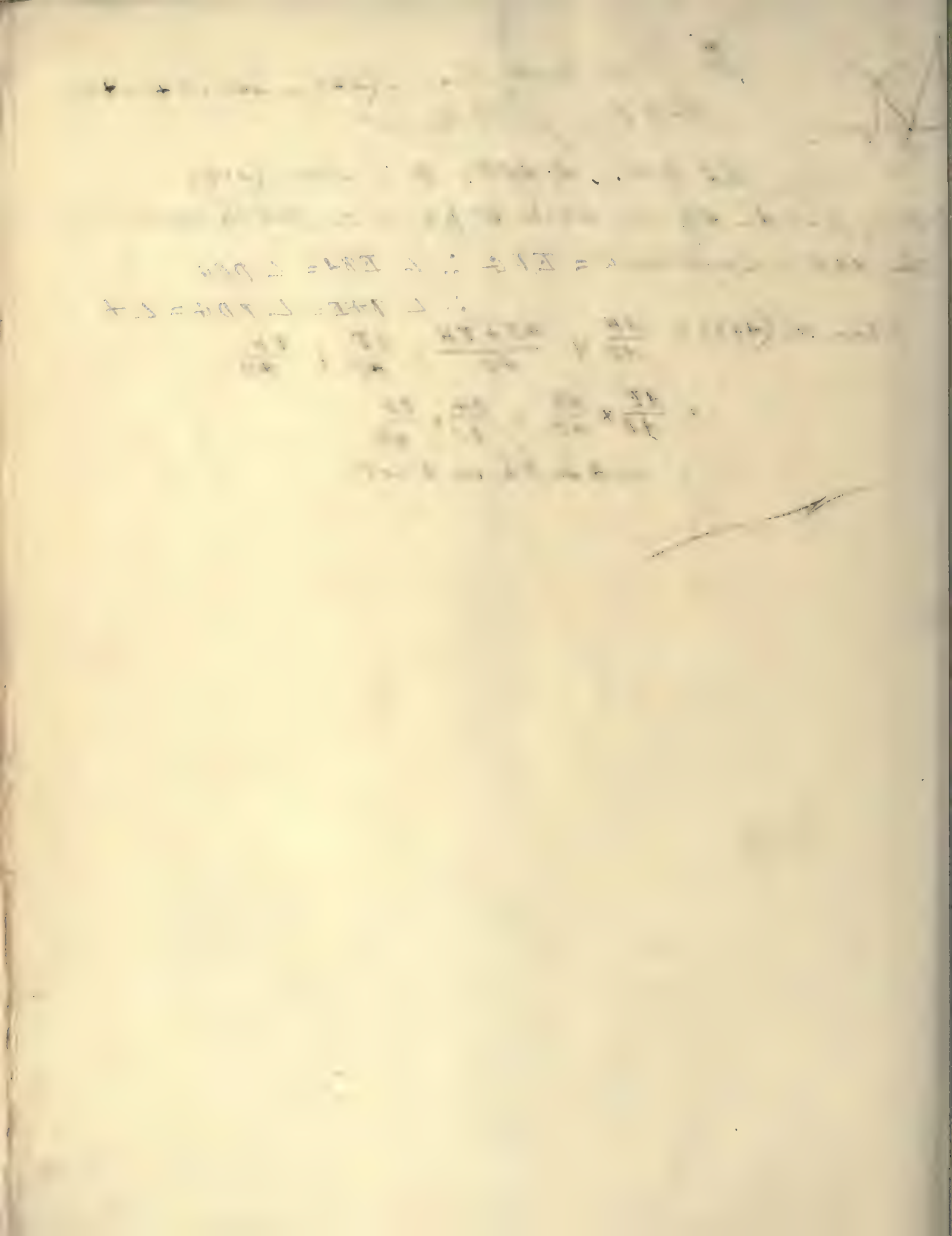
Then ABD is a right angle $= EBG$ $\therefore \angle EBA = \angle DBG$

$\therefore \angle BAE = \angle BDG = \angle A$.

Now $\sin(A+B) = \dfrac{AH}{AD} = \dfrac{AE+EH}{AD} = \dfrac{AE}{AD} + \dfrac{EH}{AD}$

$= \dfrac{AE}{AB} \times \dfrac{AB}{AD} + \dfrac{BG}{BD} \times \dfrac{BD}{AD}$

$= \cos A \sin B + \sin A \cos B$.



The angles of a plane Δ form a geometrical prog. whose common ratio is $\frac{1}{2}$ shew that the greatest side =

$2 \text{ perim} \left\{ \sin 12^\circ\, 51' 9\tfrac{5}{7}'' \right\}$

Let $2s$ = perim –

a, b, c the sides opposite to $\theta, \frac{\theta}{2}, \frac{\theta}{4}$ respectively.

then $2s = a + b + c = a \dfrac{\sin\theta + \sin\frac{\theta}{2} + \sin\frac{\theta}{4}}{\sin\theta}$

But $\sin\theta + \sin\frac{\theta}{2} + \sin\frac{\theta}{4} = 2\left\{ \cos\frac{\theta}{2}\cos\left(\frac{\theta}{4} + \frac{\theta}{8}\right) + \cos\frac{\theta}{4}\left(\cos\frac{\theta}{2} + \frac{\theta}{8}\right) \right\} + \sin\frac{\theta}{4}$

$= 4\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8} - 2\sin\frac{\theta}{8}\sin\left(\frac{\theta}{2} + \frac{\theta}{4}\right) + \sin\frac{\theta}{4}$

$= 4\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}$ since $\sin\left(\frac{\theta}{2} + \frac{\theta}{4}\right) = \cos\frac{\theta}{8}$

$\therefore s = a \left\{ \dfrac{4\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}}{8\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\sin\frac{\theta}{8}} \right.$

or $a = 2s \sin\frac{\theta}{8} = 2s \sin 12^\circ\, 51' 2\tfrac{5}{7}''$.

$\cos d = \cos^2\phi \cos^2\psi + \sin^2\phi \sin^2\psi \cos i$ — (1)

$\cos\delta = \dfrac{\cos i - \cos\phi \cos\psi}{\sin\phi \sin\psi}$ — (2)

$\cos\psi = \cos\theta \sin\phi \sin i + \cos\phi \cos i$ — (3)

Now that $\cos d = \cos 2i + 1 (\sin\phi \sin i \sin\theta)^2$

Substitute in (1) for $\cos\delta$ from (2)

$\cos d = (2\cos^2\phi - 1)(2\cos^2\psi - 1) + 4\cos\phi\cos\psi(\cos i - \cos\phi\cos\psi)$

$= 1 - 2\cos^2\phi - 2(\cos^2\psi - 2\cos\phi\cos\psi\cos i)$

Transpose (2) and square both sides.

$\cos^2\psi - 2\cos\phi\cos\psi\cos i = \cos^2\theta \sin^2\phi \sin^2 i - \cos^2\phi\cos^2 i$

$\therefore \cos d = 1 - 2\cos^2\phi + 2\cos^2\phi\cos^2 i - 2\cos^2\theta \sin^2\phi \sin^2 i$

$= 1 - 2\cos^2\phi \sin^2 i - 2\sin^2\phi \sin^2 i + 2(\sin\theta \sin\phi \sin i)^2$

$= 1 - 2\sin^2 i + 2(\sin\theta \sin\phi \sin i)^2$

$= \cos 2i + 2(\sin\theta \sin\phi \sin i)^2$.

If $\tan\frac{\alpha}{2} = \tan^2\frac{\beta}{2}$

$\tan\beta = 2\tan\theta$ } θ is an arithmetic mean between α & β.

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} + \frac{2\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}}}{1-\frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} \times \frac{2\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}}}$$

$$= \frac{2\tan^3\frac{\beta}{2}\left(1-\tan^2\frac{\beta}{2}\right) + 2\tan\frac{\beta}{2}\left(1-\tan^6\frac{\beta}{2}\right)}{\left(1-\tan^6\frac{\beta}{2}\right)\left(1-\tan^2\frac{\beta}{2}\right) - 4\tan^4\frac{\beta}{2}}$$

for $\tan\frac{\alpha}{2} = \tan^2\frac{\beta}{2}$

$$= \frac{2\tan\frac{\beta}{2}\left(1+\tan^2\frac{\beta}{2}\right)^2\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+\tan^2\frac{\beta}{2}\right)^2\left(1-3\tan^2\frac{\beta}{2}+\tan^4\frac{\beta}{2}\right)} = \frac{2\tan\frac{\beta}{2}\left(1-\tan^2\frac{\beta}{2}\right)}{1-3\tan^2\frac{\beta}{2}+\tan^4\frac{\beta}{2}}$$

$$= \frac{\frac{2\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}}}{1-\frac{\tan^2\frac{\beta}{2}}{1-3\tan^2\frac{\beta}{2}+\tan^4\frac{\beta}{2}}}$$

$$= \frac{\tan\beta}{1-\frac{1}{4}\left(\tan^2\beta\right)}$$

But $\tan\beta = 2\tan\theta$

$$\therefore \frac{\tan\beta}{1-\frac{1}{4}\tan^2\beta} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$$

Therefore $\alpha+\beta = 2\theta$, or θ is an arithmetic mean between α & β.

Let h = circular measure of an arc of 10"

Then $\sin(\theta+h) - \sin\theta = \sin\theta\cos h + \cos\theta\sin h - \sin\theta = \cos\theta\sin h - \sin\theta(1-\cos h)$

$= \cos\theta\sin h\left(1 - \tan\theta\tan\frac{h}{2}\right)$

Now if θ be not very large $\tan\theta$ may be neglected in comparison with unity and we have

$$\sin(\theta+h) - \sin\theta = h\cos\theta. \quad = D \text{ the difference if calculated for every 10"}$$

Similarly if k be a small arc containing n seconds we have

$$\sin(\theta+k) - \sin\theta = k\cos\theta = \frac{n}{10}D \Big\} \text{ for } \frac{k}{h} = \frac{n}{10}$$

Therefore if A be the degrees seconds which θ contains we have

$$\sin(A+n) - \sin A = \frac{n}{10}D.$$

Similarly on the same supposition we have

$$\cos(\theta \mp k) - \cos\theta = \cos\theta\cos k + \sin\theta\sin k - \cos\theta$$

$$= \sin\theta\sin k - \cos\theta(1-\cos k) = \sin\theta\sin k\left(1 - \cot\theta\tan\frac{k}{2}\right)$$

Now if θ be not very small, $\cot\theta$ may be neglected in comparison with unity, and we have with sufficient accuracy

$$\cos(\theta - h) - \cos\theta = h\sin\theta = D \quad \therefore \sin\theta = \frac{D}{h}$$

$$\cos(\theta - k) - \cos\theta = k\sin\theta = \frac{k}{h}D = \frac{n}{10}D$$

and as before we have

$$\cos\left(A - k\right) - \cos A = \frac{n}{10}D$$

$$\tan(\theta+h)-\tan\theta = \frac{\sin(\theta+h)}{\cos(\theta+h)} - \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta\cos\theta\cos h + \cos^2\theta\sin h - \sin\theta\cos(\theta+h)}{\cos(\theta+h)\cos\theta}$$

$$= \frac{\sin\theta\cos\theta\cos h + (\cos^2\theta+\sin^2\theta)\sin h - \sin\theta\cos\theta\cos h}{\cos^2\theta\cos h - \cos\theta\sin\theta\sin h}$$

$$= \frac{\sin h}{\cos^2\theta\cos h(1-\tan\theta\tan h)} = \frac{\sec^2\theta\tan h}{1-\tan\theta\tan h}$$

Now if θ be not very large $\tan\theta$ may be neglected in the denominator and for practical purposes the expression becomes sufficiently exact if one day

$$\tan(\theta+h)-\tan\theta = h\sec^2\theta = D \quad \therefore \sec^2\theta = \frac{D}{h}$$

also $\tan(\theta+h)-\tan\theta = h\sec^2\theta = \frac{h}{k}D.$

Therefore on the same supposition as before we have

$$\tan(\theta+n)-\tan(\theta) = \frac{n}{10}D.$$

Similarly $\cot(\theta-h)-\cot\theta = \frac{\cos(\theta-h)}{\sin(\theta-h)} - \frac{\cos\theta}{\sin\theta} = \frac{\cos(\theta-h)\sin\theta - \sin(\theta-h)\cos\theta}{\sin(\theta-h)\sin\theta}$

$$= = \frac{\cos\theta\sin\theta\cos h + \sin^2\theta\sin h - \sin\theta\cos\theta\cos h + \cos^2\theta\sin h}{\sin^2\theta\cos h - \cos\theta\sin\theta\sin h}$$

$$= \frac{\sin h}{\sin^2\theta\cos h(1-\cot\theta\tan h)} = \frac{\operatorname{cosec}^2\theta\tan h}{1-\cot\theta\tan h}$$

$$= h\operatorname{cosec}^2\theta \text{ when } \theta \text{ be very small.}$$

$$\therefore \cot(\theta-h)-\cot\theta = h\operatorname{cosec}^2\theta = D \text{ or } \operatorname{cosec}^2\theta = \frac{D}{h}$$

$$\cot(\theta-h)-\cot\theta = h\operatorname{cosec}^2\theta = \frac{h}{k}D$$

$$\cot(A-n)-\cot A = \frac{n}{10}D.$$

in the same supposition we have

$$\sec(\theta+k)-\sec\theta = \frac{1}{\cos(\theta+k)} - \frac{1}{\cos\theta} = \frac{\cos\theta - \cos(\theta+k)}{\cos(\theta+k)\cos\theta}$$

$$= \frac{\cos\theta - \cos\theta\cos k + \sin\theta\sin k}{\cos^2\theta\cos k - \cos\theta\sin\theta\sin k} = \frac{\sin\theta\sin k\left(1+\cot\theta\tan\frac{k}{2}\right)}{\cos^2\theta\cos k\left(1-\tan\theta\tan\frac{k}{2}\right)}$$

Now if k being very small neither $\tan\theta$ or $\cot\theta$ be very large, that is if θ be not near n 90. we may omit both $\cot\theta \times \tan\frac{k}{2}$, and the expression becomes

$$\sec(\theta+k)-\sec\theta = k\tan\theta\sec\theta = D \therefore \tan\theta\sec\theta = \frac{D}{k}$$

also $\sec(\theta+k)-\sec\theta = k\tan\theta\sec\theta = \frac{k}{k}D = \frac{n}{10}D$

$\therefore$ in the same supposition as before we have

$$\sec(A+n)-\sec A = \frac{n}{10}D.$$

Lastly we have

$$\operatorname{cosec}(\theta-k)-\operatorname{cosec}\theta = \frac{1}{\sin(\theta-k)} - \frac{1}{\sin\theta} = \frac{\sin\theta - \sin\theta\cos k + \cos\theta\sin k}{\sin\theta(\sin\theta-\theta+k)}$$

$$= \frac{\cos\theta\sin k + \sin\theta(1-\cos k)}{\sin^2\theta\cos k - \sin\theta\cos\theta\sin k} = \frac{\cot\theta\sin k\left\{1+\tan\theta\tan\frac{k}{2}\right\}}{\sin^2\theta\cos k\left\{1-\cot\theta\tan\frac{k}{2}\right\}}$$

and as before unless $\tan\theta$ or $\cot\theta$ be very large the part in a bracket may be omitted and we have for practical purposes

$$\operatorname{cosec}(\theta-k)-\operatorname{cosec}\theta = k.\operatorname{cosec}\theta.\cot\theta = D \therefore \operatorname{cosec}\theta\cot\theta = \frac{D}{k}$$

$$\operatorname{cosec}(\theta-n)-\operatorname{cosec}\theta = n\operatorname{cosec}\theta\cot\theta = \frac{n}{k}D = \frac{n}{10}D$$

and $\operatorname{cosec}(A+n)-\operatorname{cosec}A = \frac{n}{10}D.$

Assume $\sin(\theta+k) - \sin\theta = k\cos\theta$

Then $\log\sin(\theta+k) = \log\sin\theta + \log\{1 + k\cot\theta\}$

$\log\sin(\theta+k) - \log\sin\theta = \text{a } k\cot\theta \text{ nearly} = \delta \text{ or a } \cot\theta = \frac{\epsilon}{k}$

$\log\sin(\theta+k) - \log\sin\theta = \frac{k}{k}\delta$

$\therefore \log\sin(A+n) - \log\sin A = \frac{n}{10}\delta$

Now since the expression $\sin(\theta+k) - \sin\theta = k\cos\theta$ fails if θ be large, and $\log\sin(\theta+k) - \log\sin\theta$ involves the same quantity, it will fail under the same limitations.

$$\cos(\theta-k) - \cos\theta = k\sin\theta$$

$\log\cos(\theta-k) - \log\cos\theta = \log\{1 + k\tan\theta\}$

$\log\cos(\theta-k) - \log\cos\theta = \text{a } k\tan\theta \text{ nearly} = \epsilon \therefore \text{a}\tan\theta = \frac{\epsilon}{k}$

$\log\cos(\theta-k) - \log\cos\theta = \frac{k}{k}\delta$

$\log\cos(\theta-k) - \log\cos\theta = \frac{11}{10}\delta.$

$\tan(\theta+k) - \tan\theta = k\sec^2\theta$

$\log\tan(\theta+k) - \log\tan\theta = \log\{1 + k\sec\theta\operatorname{cosec}\theta\}$

$\log\tan(\theta+k) - \log\tan\theta = \text{a } k.\sec\theta\operatorname{cosec}\theta = \delta \therefore \text{a}\sec\theta\operatorname{cosec}\theta = \frac{\delta}{k}$

$\log\tan(\theta+k) - \log\tan\theta = \frac{k}{k}.\delta \quad \therefore \log\tan(A+n) - \log\tan A = \frac{n}{10}\delta$

$$\operatorname{cosec}(\theta-k) - \operatorname{cosec}\theta = k\cot\theta\operatorname{cosec}\theta$$

$\log\operatorname{cosec}(\theta-k) - \log\operatorname{cosec}\theta = \log\{1 + k\cot\theta\}$

$\log\operatorname{cosec}(\theta-k) - \log\operatorname{cosec}\theta = \text{a } k\cot\theta \text{ nearly} = \delta \therefore \text{a}\cot\theta = \frac{\delta}{k}$

$\log\operatorname{cosec}(\theta-k) - \log\operatorname{cosec}\theta = \frac{k}{k}\delta$

$\log\operatorname{cosec}(A-n) - \log\operatorname{cosec} A = \frac{n}{10}\delta.$

Trigonometry.

1. Find all the vals. of the $\angle \theta$ which satisfy the eqn $\tan\theta = a$.

2. Assuming the truth of the formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

when A is $> B$ and $A + B < 90°$, prove them true for all vals. of A & B.

3. Find the sine and cosine of $\frac{1}{2}$ an $\angle$ interms of the sine of the angle, and explain the reason of there being four vals. for each.

4. Prove that if $A + B + C = 180°$

(1) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

(2) $\cos A + \cos B + \cos C = 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} + 1$

(3) $\cot A + \cot B + \cot C = \cot A \cot B \cot C + \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$

(4) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

5. If $\cos d = \cos 2\varphi \cos 2\psi + \sin 2\varphi \sin 2\psi \cos\delta$

$\sin\varphi \sin\psi \cos\delta + \cos\varphi \cos\psi = \cos\angle$

& $\cos\psi = \cos\theta \sin\varphi \sin\angle + \cos\varphi \cos\angle$

(Hint: a

Prove that

$$\cos\alpha = \cos 2\lambda + 2\sin^2\varphi \sin^2\lambda \sin^2\theta$$

6. Prove that if a be the base, and h the altitude of the $\triangle ABC$ of any $\triangle$ is given

$$= \frac{a^2 + b^2 + c^2}{4(\cot A + \cot B + \cot C)}$$

7. Prove that

$$\tan n\varphi = \frac{\sin\varphi + \sin 3\varphi + \&c. + \sin(2n-1)\varphi}{\cos\varphi + \cos 3\varphi + \&c. + \cos(2n-1)\varphi}$$

To find $\sum^{n} u_r$ where $u_r = \dfrac{\sec \overline{2r-1}\,\theta \sec \overline{2r+1}\,\theta}{\operatorname{cosec} 2r\theta}$

$$2u_r \sin\theta = \frac{2\sin 2r\theta \sin\theta}{\cos \overline{2r-1}\,\theta \cos \overline{2r+1}\,\theta}$$

$$= \frac{\cos \overline{2r-1}\,\theta - \cos \overline{2r+1}\,\theta}{\cos \overline{2r-1}\,\theta \cos \overline{2r+1}\,\theta}$$

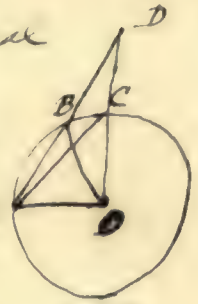
$$= \sec \overline{2r+1}\,\theta - \sec \overline{2r-1}\,\theta$$

$$2\sin\theta \sum^{n} u_r = \sum^{n} \sec \overline{2r+1}\,\theta - \sum^{n} \sec \overline{2r-1}\,\theta$$

$$= \sec \overline{2n+1}\,\theta - \sec\theta$$

$$\sum^{n} u_r = \frac{2\sin \overline{n+1}\,\theta \sin n\theta}{\sin 2\theta \cos \overline{2n+1}\,\theta}$$

AB & AC are the chords of arcs of 60° each ∴ in a circle whose centre is O, then that if AB and AC be produced to meet in D, the arc whose chord = DC has its cosine and chord equal



For BO. then AB = rad of 3 = BC. = 1 suppose
Also angle ABO = interior angles BDO, BOC,
But ABO contains 60° and BOC contains 30°
by construct: ∴ BDO contains 30°
Therefore BD = BC = OC.

$$\text{Now } OD = \sqrt{AD^2 - AO^2} = \sqrt{4-1} = \sqrt{3}$$

$$\text{also } OC = r = 1 \quad \therefore CD = \sqrt{3} - 1.$$

Let θ be the angle whose chord equals its cosine

$$\text{Then } 2\sin\frac{\theta}{2} = \cos\theta = 1 - 2\sin^2\frac{\theta}{2} \quad \text{or } \sin^2\frac{\theta}{2} + \sin\frac{\theta}{2} = \frac{1}{2}$$

$$\therefore \text{Completing the square } \sin\frac{\theta}{2} = \frac{1}{2}\{\sqrt{3}-1\}$$

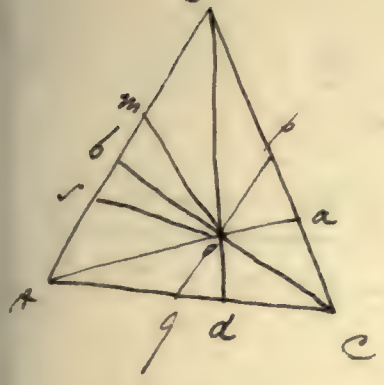
$$\therefore 2\sin^2\frac{\theta}{2} = \frac{1}{2}\{4 - 2\sqrt{3}\} = 2 - \sqrt{3}$$

$$\text{and } \cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - \{2-\sqrt{3}\} = \sqrt{3}-1 = \text{chord } \theta$$

Therefore the angle which has its cosine = to its chord, is subtended by a chord whose length is $\sqrt{3}-1$

But it was before shewn that $CD = \sqrt{3}-1$

∴ the arc subtended by CD has its chord = to its cosine.



Shew that $\frac{oa}{Aa} + \frac{od}{Ba} + \frac{oc}{bc} = 1$

Draw pq // to AB

Draw om // to BC

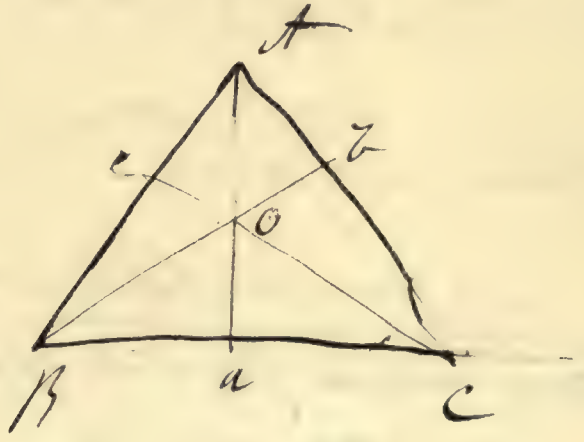
$\therefore \frac{oa}{Aa} = \frac{Bm}{AB}$

Draw os // to AC $\therefore \frac{od}{Ba} = \frac{As}{AB}$

also $\triangle$ mos is equiangular to $\triangle$ BAC

$\therefore \frac{ms}{AB} = \frac{so}{AC} = \frac{Aq}{AC} = \frac{bo}{bc}$ $\therefore \frac{bo}{bc} = \frac{ms}{AB}$

and $\frac{oa}{Aa} + \frac{od}{Ba} + \frac{ob}{bc} = \frac{Bm + As + sm}{AB} = \frac{AB}{AB} = 1$



$$\tan 3x = \frac{\sin 3x}{\cos 3x} = \frac{\sin 3x(1+2\cos 2x)}{\cos 3x(1+2\cos 2x)} = \frac{\sin 3x + 2\sin 3x\cos 2x}{\cos 3x + 2\cos 3x\cos 2x} =$$

$$\frac{\sin x + \sin 2x + \sin 5x}{\cos x + \cos 3x + \sin 5x}$$

Q.E.D.

1) $2\sin(45^\circ \pm a) = \sqrt{2}\{\sin 45^\circ \cos x \pm \cos 45^\circ \sin x\} = \sqrt{2}\{\cos x \pm \sin x\}$

$= \sqrt{2\cos^2 x} \pm \sqrt{2\sin^2 x} = \sqrt{1+\sin 2x} \pm \sqrt{1-\sin^2 x}$,

3) $2\cos(45^\circ \pm a) = \sqrt{1+\cos 2x} \mp \sqrt{1-\sin 2x}$ in a similar manner.

4) $\frac{\sin 2a + 2\sin^2 a - 2\sin a}{1-\cos a} = \frac{\sin a\cos a + \sin^2 a - \sin a}{\frac{1-\cos a}{2}} =$

$\frac{\sin^2 a - (1-\cos a)\sin a}{\sin^2 \frac{a}{2}} = \frac{4\sin^2\frac{a}{2}\cos^2\frac{a}{2} - 2\sin^2\frac{a}{2} + 2\sin\frac{a}{2}\cos\frac{a}{2}}{\sin^2\frac{a}{2}} =$

$4\cos^2\frac{a}{2} - 4\sin\frac{a}{2}\cos\frac{a}{2} = 4\cos\frac{a}{2}\left\{\cos\frac{a}{2} - \sin\frac{a}{2}\right\}$

5) $\frac{\tan a - \sin a}{\sin^2 a} = \frac{\frac{\sin a}{\cos a} - \sin a}{\sin^3 a} = \frac{1-\cos a}{\cos a(1-\cos^2 a)} = \frac{1}{\cos a(1+\cos a)}$

6) $\frac{\cos a + \sin a - \frac{1}{2}\sin 2a - \sin^2 x}{2\cos 2x + 2\sin^2 x + \sin x - \sin^2 x} = \frac{\cos a + \sin a + \sin a - \cos a + \sin a + \sin a}{2\cos^2 x - 2\sin^2 x + (2\sin^2 x - 1)\sin x + 2\sin x - \sin^2 x}$

$= \frac{(1-\sin a)\cos a + (1-\sin a)\sin a}{2\cos^2 a - 3\sin^2 x + (1-2\cos^2 x)\sin x + 2\sin x} = \frac{(\cos a + \sin a)(1-\sin a)}{2\cos^2 x + (1-\sin a) + 3\sin a(1-\sin a)}$

$= \frac{\cos a + \sin a}{3\sin a + 2\cos^2 a}$

Trigonometrical Reductions. Form. Add.

$\angle 37$

$1 + 2\cos 74$

$\cos 3a \cos 5a = \cos^2 4a - \sin^2 a$

$1^{st}.\ \cos 5a = \cos(3a+2a) = \cos 3a \cos 2a - \sin 3a \sin 2a$

$\therefore \cos 3a \cos 3a = \cos^2 3a \cdot \cos 2a - \sin 3a \cos 3a \sin 2a$

$= \cos^2 3a (2\cos^2 a - 1) - 2\sin 3a \cos 3a \sin a \cdot \cos a$

$= \cos^2 3a \cdot \cos^2 a - 2\sin 3a \cos 3a \sin a \cos a + \cos^2 3a (\cos^2 a - 1)$

$= \cos^2 3a \cos^2 a - 2\sin 3a \cos 3a \sin a \cos a + \sin^2 a \sin^2 3a - \cos^2 3a$

$\cos 5a \cos 3a + \sin^2 a = \cos^2 3a \cos^2 a - 2\sin 3a \cos 3a \sin a \cos a + \sin^2 a \sin^2 3a$

$= (\cos 3a \cos a - \sin 3a \sin a)^2$

$= \cos^2 4a$

$\cos 5a \cdot \cos 3a = \cos^2 4a - \sin^2 a$

To find the 3 cube roots of 1

$\text{Let } (\cos\theta + \sqrt{-1}\sin\theta)^{\frac{1}{3}} = \cos\frac{\theta}{3} + \sqrt{-1}\sin\frac{\theta}{3} = \cos\frac{2\pi \pm \theta}{3} + \sqrt{-1}\sin\frac{2\pi \pm \theta}{3}$

$\text{Let } \theta = 0$

$\therefore (\cos\theta + \sqrt{-1}\sin\theta)^{\frac{1}{3}} = 1^{\frac{1}{3}}$ for $\sin\theta = 0$ when $\theta = 0$ and $\cos\theta = 1$

$\therefore 1^{\frac{1}{3}} = 1 = \cos 120^\circ + \sqrt{-1}\sin 120^\circ = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$

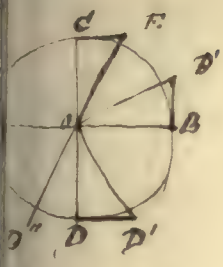
$= \cos\frac{4\pi \pm \theta}{3} + \sqrt{-1}\sin\frac{4\pi \pm \theta}{3}$

$= \cos 240^\circ + \sqrt{-1}\sin 240^\circ = -\frac{1}{2} - \frac{\sqrt{-3}}{2}$

$\therefore$ The 3 cube roots are: $1, -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}$

Prove that $\cot\left\{(4n+3)\frac{\pi}{2} \pm A\right\} = \mp \tan A$

$\cot\left\{(4n+3)\frac{\pi}{2} \pm A\right\} = \cot\frac{3\pi}{2} \pm A.$



Let ACBD be a circle, and AB, CD two diameters, cutting one another at right angles. If the point O make $\angle DOD' = A$. Then $\angle DOD'' = \left(\frac{3\pi}{2} + A\right)$. From D draw DD' at right angles to OD and it will be the $\cot\left(\frac{3\pi}{2}+A\right)$ which is negative

Next make the $\angle BOB'$ as in last fig. and $\angle BD'$ at right angles to OB. Then BD' = tan BOB which is positive

But as in the last case it may be shewn that BB' = DD' with its sign changed: therefore

$$\cot\tan\left(\frac{3\pi}{2}+A\right) = \cot\left\{(4n+3)\frac{\pi}{2}+A\right\} = -\tan A$$

Also $\cot\left\{(4n+3)\frac{\pi}{2} - A\right\} = \cot\left(\frac{3\pi}{2} - A\right)$

and $\tan\left(\frac{3\pi}{2} - A\right)$ shall equal $\mp \tan A$

For the same angles being constructed as in the last figure and OD'' being produced to E and CE drawn at right angles to OC it will be $\cot\left(\frac{3\pi}{2} - A\right)$ which is positive

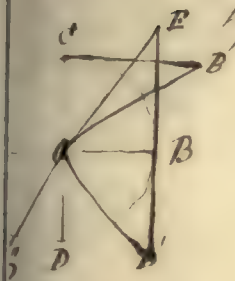
Which may be shewn that CE = BD' and they are both positive: therefore $\cot\left(\frac{3\pi}{2} - A\right) = \cot\left\{(4n+3)\frac{\pi}{2} - A\right\} = \tan A$

And generally as before

$$\cot\left\{(4n\pm 3)\frac{\pi}{2} \pm A\right\} = \mp \tan A.$$

$\tan\left(\{4n+3\}\frac{\pi}{2} \pm A\right) = \mp \cot A$

$\tan(4n+3)\frac{\pi}{2} = \tan\frac{3\pi}{2}$



Let ACBD be a circle, and draw the diameter CD & thro' the diameter AB at right angles. Then $\angle BCAD = \frac{3\pi}{2}$ At the point O in the straight line DO make any angle DOD' which let be the angle A and from B the extremity of the diameter AB draw BD' at right angles to AB. Then BD' is $\tan\left(\frac{3\pi}{2} + A\right)$

Next at the point O in the straight line OB make $\angle BOB' =$ to DOD' and draw CB' at right angles to CD & produced it to B'. Then CB' = cotangent of A

Now since $\angle COB = \angle BOD$

and $\angle BOB' = \angle DOD''$

$\therefore \quad \angle BOD' = \angle COB'$

also $\angle B'CO = \angle DBO$ for each of them is a right angle

and side OC = side OB being radii of the same circle

Therefore by one side CB' = side BD'

But BB = $\tan\left(\frac{3\pi}{2} + A\right)$ which is negative

and CB = cot A which is positive

consequently $\tan\left(\frac{3\pi}{2} + A\right) = \tan\left\{(4n+3)\frac{\pi}{2} + A\right\} = -\cot A$

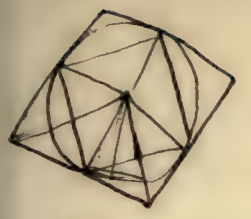
Also $\tan\left(\frac{3\pi}{2} - A\right) =$ cot A, for in the last figure make $\angle DOD'' = \angle DOD$ & produce D''O till it meets D'B produced in E. Then EB = tang $\frac{3\pi}{2} - A$ which is positive

But as before it may be shewn that CB' = EB and they are both positive

Therefore $\tan\left(\frac{3\pi}{2} - A\right) = \tan\left\{(4n+3)\frac{\pi}{2} - A\right\} = \cot A$

and generally as above —

$\tan\left\{(4n+3)\frac{\pi}{2} \pm A\right\} = \mp \cot A.$



Trigonometry. Bird p. 238
Ex 41.

$n \quad \frac{180}{n}$

$r \cdot \cos \frac{180}{n} \times r \sin \frac{180}{n}$

$= nr^2 \cdot \cos \frac{180}{n} \times \sin \frac{180}{n} = a$

$nr^2 \cdot \tan \frac{180}{n} = b$

$\frac{nr^2 \cdot \sin A \cos A = a}{nr^2 \cdot \frac{\sin A}{\cos A} = b} = \frac{\cos A = a}{\frac{1}{\cos A} = b} \therefore \cos^2 A = \frac{a}{b}$

$\cos A = \sqrt{\frac{a}{b}}$

$\sin A = \sqrt{1 - \cos^2 A}$

$= \sqrt{\frac{b-a}{b}}$

$2 \sin^2 \frac{A}{2} = 1 - \cos A$

$= 1 - \sqrt{\frac{a}{b}} = \frac{\sqrt{b} - \sqrt{a}}{\sqrt{b}}$

$\sin^2 \frac{A}{2} = \frac{\sqrt{b} - \sqrt{a}}{2\sqrt{b}}$

$\sin \frac{A}{2} = \left\{ \frac{b^{\frac{1}{2}} - a^{\frac{1}{2}}}{2b^{\frac{1}{2}}} \right\}^{\frac{1}{2}} = \sin \frac{180^\circ}{n}$

similarly $\cos \frac{A}{2} = \left(\frac{b^{\frac{1}{2}} + a^{\frac{1}{2}}}{2b^{\frac{1}{2}}} \right)^{\frac{1}{2}} = \cos \frac{180^\circ}{2n}$

Now area of inscribed polygon of n sides $= n \left\{ r^2 \cos \frac{180^\circ}{n} \cdot \sin \frac{180^\circ}{n} \right\}$

$= \frac{1}{2} nr^2 \left\{ \frac{b-a}{4b} \right\}^{\frac{1}{2}}$ (1)

Now $nr^2 \sin A \cos A = a$

$nr^2 \cdot \sqrt{\frac{b-a}{b}} \sqrt{\frac{a}{b}} = a$

$nr^2 \sqrt{\frac{ab - a^2}{b^2}} = a$

$nr^2 = \frac{ab}{\sqrt{ab - a^2}}$

$\frac{1}{2} nr^2 = \frac{ab}{2\sqrt{ab - a^2}}$

By substituting in (1) the area of the inscribed polygon becomes

$\frac{ab}{2\sqrt{ab - a^2}} \cdot \frac{(b-a)^{\frac{1}{2}}}{(4b)^{\frac{1}{2}}} = \left(\frac{a^2b^3 - a^3b^2}{4ab^2 - 4a^2b} \right)^{\frac{1}{2}} =$

$= \left(\frac{ab^2 - a^2b}{4b - 4a} \right)^{\frac{1}{2}}$

$= \sqrt{ab}$

Ag a = area of circumscribing polygon

$= \frac{1}{2} nr^2 \left\{ \tan \frac{180^\circ}{n} \right\}$

$= \frac{1}{2} nr^2 \left\{ \frac{b^{\frac{1}{2}} - a^{\frac{1}{2}}}{b^{\frac{1}{2}} + a^{\frac{1}{2}}} \right\}^{\frac{1}{2}}$

$= \frac{1}{2} \cdot \frac{(ab)^{\frac{1}{2}}}{\sqrt{ab - a^2}} \cdot \left(\frac{b^{\frac{1}{2}} - a^{\frac{1}{2}}}{b^{\frac{1}{2}} + a^{\frac{1}{2}}} \right)^{\frac{1}{2}} = 2 \left\{ \frac{a^2b^3 - a^3b^2}{ab^2 - a^2b + ab^2 - a^3} \right\}^{\frac{1}{2}}$

$= 2 \left\{ \frac{ab^3 - a^2b^2}{b^2 - a^2} \right\}^{\frac{1}{2}} = \left(\frac{ab^2}{b+a} \right)^{\frac{1}{2}} = b\sqrt{\frac{a}{a+b}}$

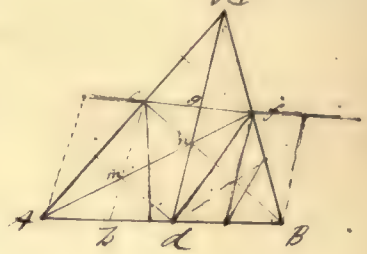
Therefore the area of inscribed polygon $= \sqrt{ab}$

area of circumscribed polygon $= 2b\sqrt{\frac{a}{a+b}}$.

$\sin \frac{A}{2} = \sqrt{\frac{a}{2}} \cdot \frac{1}{2}$ $\quad \sin \frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}}$

$= 1 -$

Given one angle of a triangle, and the straight lines drawn from each of the other angles to bisect the opposite sides. to find the sides of the triangle



Let $\angle B = 60°$ and $Cd = a$: then if e, f the middle points of the opposite sides are joined $eg = gf = fh = hd$
$\therefore fn = mn = ne$

Let $Af = 3a \therefore fn = a$
Let $Cd = 2b \therefore dn = b$
$Be = 3c \therefore en = c$
} Given

Also let $AC = x$, $CB = y$. $AB = z$.

Then $\frac{1}{2}x = \frac{x}{2}$. and in the $\triangle$ endm. $en = c$. $nd = b$. $nm = a$

$$\cos \angle men = \frac{a^2 + c^2 - b^2}{2ac} = \frac{(3c)^2 + \left(\frac{3b}{2}\right)^2 - \left(\frac{3}{4}x\right)^2}{2 \cdot 3c \times \frac{3b}{2}}$$

$\angle B$

$$\frac{a^2 + c^2 - b^2}{2ac} = \frac{9c^2 + \frac{9b^2}{4} - \frac{9x^2}{16}}{9cb}$$

$$9cb(a^2 + c^2 - b^2) = 2ac\left(9c^2 + \frac{9b^2}{4}\right) - \frac{9acx^2}{8}$$

$$x^2 = 8\left\{\frac{18ac\left(c^2 + \frac{b^2}{4}\right) - 9cb(a^2 + c^2 - b^2)}{9ac}\right.$$

$$= 8\left\{\frac{2c\left(c^2 + \frac{b^2}{4}\right) - b(a^2 + c^2 - b^2)}{a}\right.$$

$$x = \sqrt{8\left\{\frac{2c\left(\frac{4c^2 + b^2}{4}\right) - b(a^2 + c^2 - b^2)}{a}\right.}.$$

Let $\angle A = 60°$. $\frac{x}{2} \angle A = 3b \angle - dCA$

$$\frac{1}{2}\sqrt{3}\,\frac{x}{2} = 3b \angle - dCA$$
$$\frac{x}{4}\sqrt{3} = 3b \angle - dCA$$
$$\frac{x\sqrt{3}}{12a} = \angle - dCA \quad \text{and} \quad 120° - dCA = CdA = \text{I suppose}$$

Then $\alpha \angle - bc = Cd$ a rw.

$$\frac{\alpha}{2}\sqrt{3} = 3b \text{ a rw}$$
$$x\sqrt{3} = 6b \text{ a rw}$$
$$\alpha = \frac{6b}{\sqrt{3}} \text{ a rw}$$

Trigon. Book. P. 336.

11. Given area of right angled triangle and rad of inscribed ○. to find the sides.

Let p = area. r = rad of circle. $x+y$ = hypoth. $x+r$ & $y+r$ = sides

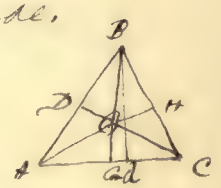
$$p = \frac{(x+r)(y+r)}{2} = \frac{xy+xr+r^2+ry}{2} \quad \therefore x(y+r) = 2p - r^2 - ry$$

$$x = \frac{2p - r^2 - ry}{y+r} \quad (1)$$

Alto. $(x+y)^2 = (x+r)^2 + (y+r)^2$

or $x = \frac{r^2+ry}{y-r}(11) = \frac{2p-r^2-ry}{y+r}$ from which $y^2 + (r^2+p)y = -pr$. and y may be discovered

12. Given angles and radius of inscribed circle to find sides.



Let $OD = r$. angle $A = 60°$ $\therefore \angle DAO = 30°$

Then $DA = OD \cot OAD = r \cot 30° = r\sqrt{3}$

$\therefore AD$ or $AG = r\sqrt{3}$

Similarly CG or $CH = r \cot \frac{C}{2}$ and BD or $BH = r \cot \frac{B}{2}$.

13. Express the area of a triangle in terms of the inscribed circle, and the angles.

Draw Bd at right angles to AC

Then area of $\Delta = \frac{AC \times Bd}{2}$.

Now $BH = r \cot \frac{B}{2}$, $HC = r \cot \frac{C}{2} = CG$ and $AG = r \cot \frac{A}{2}$

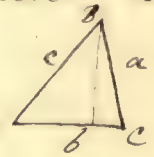
$\therefore (BH + HC) = BC = r\left(\cot \frac{B}{2} + \cot \frac{C}{2}\right)$ and $AC = r\left\{\cot \frac{A}{2} + \cot \frac{B}{2}\right)$

and $Bd = BC \sin C \quad \therefore r \sin C \left\{\cot \frac{B}{2} + \cot \frac{C}{2}\right) = Bd$

and area $= \frac{r \sin C \left\{\cot \frac{B}{2} + \cot \frac{C}{2}\right\} \times r \left\{\cot \frac{A}{2} + \cot \frac{B}{2}\right\}}{2}$

---

14. Given three sides of a triangle and radius of circumscribed circle to find the sides



Here denoting the diameter of the circumscribing circle by D

we have $D = \frac{a}{\sin A} \quad \therefore a = D \sin A$ or $a = 2R \sin A$

$b = 2R \sin B$

$c = 2R \sin C$

---

15. area of triangle $ABC = \frac{AC \times BB}{2}$ if BB be perpendicular to AC

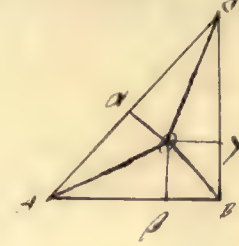
Now by last question $AC = 2R \sin B$

and $BB = BC \sin C = 2R \sin A \sin C$

$\therefore$ area $= 2R^2 \sin A \sin B \sin C$

Trigonometrical Problem.

Given the hypothenuse of a right-angled triangle and the radius of the inscribed circle, to find the sides



Let $\alpha A, \alpha B, \alpha\gamma$ = radius of the inscribed circle

Then from construction $\angle \alpha C = \gamma C$

$\alpha A = BA$

Also $BB = B\gamma = \alpha\gamma = \alpha B$ for each of the angles $\alpha B B$, ABC is a right angle, consequently $\alpha\gamma BB$ is a parallelogram

Let $\alpha C = C\gamma = x$ $\therefore$ $AC = AB = b$ if $AC = b$ and let rad $= r$

Then $AB = AB + BB = b - x + r$, and $CB = C\gamma + \gamma B = x + r$

$$\therefore b^2 = (b - x + r)^2 + (x + r)^2$$

$$= b^2 - 2bx + x^2 + r^2 + 2br - 2xr + x^2 + 2rx + r^2$$

$$0 = 2x^2 + 2r^2 - 2bx + 2br$$

$$x^2 - bx = -r(r+b)$$

$$x^2 - bx + b^2 = b^2 - r(r+b)$$

$$x - b = \pm\sqrt{b^2 - r(r+b)}$$

$$x = b \pm \sqrt{b^2 - r(r+b)}$$

Therefore $AC = b$

$$AB = r + \sqrt{b^2 - r(r+b)}$$

$$CB = b + r - \sqrt{b^2 - r(r+b)}$$

Given the perimeter and radius of the inscribed circle to find the sides.

The same construction being made as in the last for let $C\alpha = C\gamma = y$

$A\alpha = AB = z$ and $B\beta = \text{rad} = B\gamma = r$ also let perimeter $= 2a$

Then $AB = z + r$, $AC = y + z$, $CB = y + r$

Also $z + r + y + z + y + r = 2(y + z + r) = \text{perim} = 2a$ or $2y = 2a - 2(z + r)$

$$\therefore y = a - z - r \quad (I)$$

Again from right angled triangle

$$AC^2 = AB^2 + BC^2 \text{ or } (y + z)^2 = (y + r)^2 + (z + r)^2$$

Equate (I) and II

$$y^2 + 2yz + z^2 = y^2 + 2ry + r^2 + z^2 + 2rz + r^2$$

$$(z - r)2y = 2ry + r^2$$

$$z = \frac{2ry + r^2}{2y - r}$$

$$\text{and } y = \frac{r(z + r)}{z - r} \quad (II)$$

$$r(z + r) = z - r(a - z - r)$$

$$z^2 + (r - a)z + \frac{r^2 - 2ar + a^2}{4} = \frac{r^2 - 6ar + a^2}{4}$$

$$z + \frac{r - a}{2} = \frac{1}{2} \pm \frac{\sqrt{r^2 - 6ar + a^2}}{4}$$

$$z = \frac{1}{2}\{a - r \pm \sqrt{r^2 - 6ar + a^2}$$

$$AB = \frac{1}{2}\{a + r \pm \sqrt{r^2 - 6ar + a^2}$$

$$CB = \frac{1}{2}\{a - r \mp \sqrt{r^2 - 6ar + a^2}$$

$$AC = a - 2r$$

Trigonometrical Problem.

The area of a regular polygon of n sides in a circle: the area of another polygon of $3n$ sides in the same circle as $p : q$. find the value of the angle subtended by a side of each at the centre.

Let O be the centre r the radius of the circle. AB the side of the polygon.

Then is a polygon of n sides $\angle AOB = \frac{360}{n} = 2\theta$.

$\therefore \angle AOD = \frac{180}{n}$. $\therefore OD = r\cos\frac{180}{n}$ and $AD = r\sin\frac{180}{n}$

similarly in a polygon of $3n$ sides $\angle AOD = \frac{60}{n}$. and area of polygon $= nr^2 \sin\frac{180}{n}\cos\frac{60}{n}$

and as before $OD = r\cos\frac{180}{3n}$. $AD = r\sin\frac{180}{3n}$

and area of the polygon $= 3nr^2 \sin\frac{60}{n}\cos\frac{60}{n}$.

Then by question. $nr^2 \sin\frac{180}{n}\cos\frac{180}{n} : 3nr^2 \sin\frac{60}{n}\cos\frac{60}{n} :: p : q$.

Let $\frac{60}{n} = \theta$ $\therefore \sin 3\theta\cos 3\theta : 3\sin\theta\cos\theta :: p : q$

$$q\{(3\sin\theta - 4\sin^3\theta)(4\cos^3\theta - 3\cos\theta)\} = p\sin\theta\cos\theta$$

$$q\{(3 - 4\sin^2\theta)(4\cos^2\theta - 3)\} = p$$

$$16\sin^4\theta - 16\sin^2\theta = p - 3q$$

$$\sin^4\theta - \sin^2\theta + \frac{1}{4} = \frac{p + 12q}{16q}$$

$$\sin^2\theta - \frac{1}{2} = \pm\sqrt{\frac{p+12q}{16q}}$$

$$\sin\theta = \frac{1}{2}\sqrt{1 \pm \sqrt{\frac{p+12q}{16q}}} = \beta$$

and $\cos 3\theta = \cos\theta\{3 - 4\sin^2\theta\} = \sqrt{1 \pm \sqrt{\frac{p+12q}{16q}}}\left\{7 + 4\sqrt{\frac{p+12q}{16q}}\right\} = \alpha$ suppose.

Therefore a side of the polygon of n sides $= 2r\sin 3\theta$

$$= 2r\left(7 + 4\sqrt{\frac{p+12q}{16q}}\right)\sqrt{1 \mp \sqrt{\frac{p+12q}{16q}}}$$

and the angle subtended $= 6\theta$.

$\therefore \sin 6\theta = 2\sin 3\theta\cos 3\theta = 2\gamma\sqrt{1-\gamma^2}$

and the angle subtended by a side of the ~~smaller~~ polygon of $3n$ sides $=$

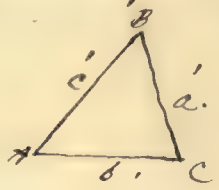
$\therefore \sin 2\theta = 2\sin\theta\cos\theta = 2\beta\sqrt{1-\beta^2}$

24

Trigonometrical Problems.

Given the perimeter of a triangle and the ratios of its angles, to find the sides

Let $a'+b'+c' = a$ and let the angles be to each other as 2. 3. 4. viz $\angle A = 2$ $\angle B = 3$ $\angle C = 4$



Then $2+3+4 = 9$ and $\frac{180^\circ \times 2}{9} = 40^\circ = \angle A$

$$\frac{180^\circ \times 3}{9} = 60^\circ = \angle B$$

$$\frac{180^\circ \times 4}{9} = 80^\circ = \angle C$$

Then $c' \sin A = a' \sin C = a' \sin 2A = a' 2 \sin A \cos A$

$\therefore c' = a' \cos A = a' \cos 40^\circ$

also $b' \sin A = a' \sin B = a' \frac{\sqrt{3}}{2}$

and $\sin A = \frac{a'\sqrt{3}}{2b}$ $\therefore b' = \frac{a'\sqrt{3}}{2(\sin 40^\circ)}$

$c' \sin B = b \sin C$

$\frac{c'\sqrt{3}}{2} = b \sin A \sqrt{1-\sin^2 A}$

$\frac{c'\sqrt{3}}{2} = b \cdot \frac{a'\sqrt{3}}{b} \sqrt{1 - \frac{3a'^2}{4b^2}}$

$\frac{3c'^2}{4} = 3a'^2 \left(1 - \frac{3a'^2}{4b^2}\right)$

$\therefore c' = \frac{4}{3}\left\{3a'^2 - \frac{3a'^4}{4b^2}\right\} = 4a'^2 - 3\frac{a'^4}{b^2}$

$c' + b' + a' = a'\left(\cos 40^\circ + \frac{\sqrt{3}}{2\sin 40^\circ} + 1\right) = a$

and $\sin 60^\circ$ being found from the tables, whence the solution is effected.

$\frac{3c'^2}{4} = 4b^2 \sin^2 A (1 - \sin^2 A)$

$3c'^2 = 16b^2 \sin^2 A - 16b^2 \sin^4 A$

$16b^2 \sin^4 A - 16b^2 \sin^2 A = -3c'^2$

$\sin^4 A - \sin^2 A = -\frac{3}{16}\left(\frac{c'}{b'}\right)^2$

$\sin^4 A - \sin^2 A + \frac{1}{4} = \frac{1}{4} - \frac{3}{16}\left(\frac{c'}{b'}\right)^2$

$\sin^2 A - \frac{1}{2} = \sqrt{\frac{1}{4} - \frac{3}{16}\left(\frac{c'}{b'}\right)^2}$

$\sin A = \pm\sqrt{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{3}{16}\left(\frac{c'}{b'}\right)^2}}$

If a circle be described about a triangle, find the distances of the bisections of the sides from the circumference

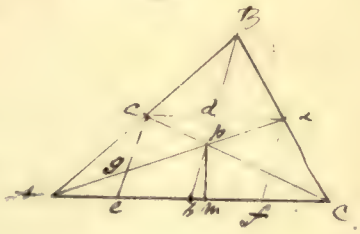
Assuming for data the radius of the circumscribed circle (R) and the sides or angles of the Δ we have

$$cO = R \sin \frac{C}{2} \qquad R^2 - \frac{c^2}{4}$$
$$aO = R \sin \frac{A}{2} \qquad = R^2 - \frac{a^2}{4}$$
$$bO = R \sin \qquad = R^2 \quad \frac{b^2}{4}$$

Therefore the distances of

$$\left.\begin{matrix} a \\ b \\ c \end{matrix}\right\} \text{ from the circumf} = \left\{\begin{matrix} R(1-R) + \frac{a^2}{4} \\ R(1-R) + \frac{b^2}{4} \\ R(1-R) - \frac{c^2}{4} \end{matrix}\right.$$

Given the three straight lines drawn from the angles of a triangle, to bisect the opposite sides to find the sides of the triangle.



If a, c, b be the points of bisection — $Aa = ca$.
Then because the lines AB, BC are each bisected
ca is parallel to AC. also $AC = 2ac$
$\angle BCa = \angle CAb$
$\angle BAc = \angle ACA$
$\therefore$ angle cA = angle Ac

Join $cd = da$ for $Ab = bC$ Therefore also $Ae = eb = bf$ $\therefore Ag = gb = bd$
for $ge, Ab,$ of are parallels. Now the distance Aa is given $= 3a'$ suppose
$\therefore Ab = 2a'$ similarly $bC = 2b'$ and $bb = c'$ if the other lines are drawn by
$2b'$ and $3c'$ as before. If Ab be not perpendicular to AC draw bm perpendicular

~~$Bb^2 = Ab^2 + Ab^2 + 2 Ab \times bm$~~ also $Bc^2 = Ab^2 + bC^2 - 2AC \times bm$
$\therefore Bb^2 = Ab^2 + Ab^2 - 2 Ab \times bm$ $\therefore Ab^2 = Bc^2 - bC^2 + 2AC \times bm$

$\therefore Ab^2 - Bb^2 = 2Ab \times bm = Bc^2 - bC^2 + 2AC \times bm$ or by doubling the last

$4a'^2 - 4c'^2 = 2Ab \times bm = Ab^2 - bC^2 + 2AC \times bm$ (by subtraction of (1) from (1))

$4(a'^2 - b'^2) = 4Ab \times bm$

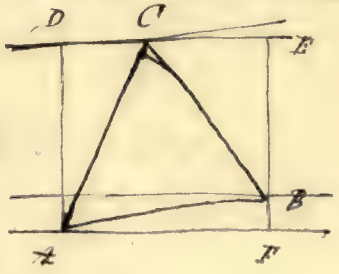
Let $Am = x + y$ $\therefore mC = a - y$
then $4a'^2 - (x+y)^2 = 4b'^2 - (a-y)^2$
$4a'^2 - x^2 - 2xy - y^2 = 4b'^2 - a^2 + 2ay - y^2$
$4(a'^2 - b'^2) = 4ay$ (1)

$y = \sqrt{2a'^2 - 2b'^2 - c'^2}$
$\therefore 4(a'^2 - b'^2) = 4a\{\sqrt{2a'^2 - 2b'^2 - c'^2}\}$

$\sin(bm)^2 = 4a'^2 - (x+y)^2 = 4a'^2 - x^2 - 2xy - y^2$
$b'^2 = (bm)^2 + (Cm)^2$ or $c^2 = x^2 + 4b'^2 - a^2 - 2ay - y^2$

$x = \frac{a^2 - b'^2}{\sqrt{2a'^2 - 2b'^2 - c'^2}}$

If an equilateral triangle have its angular points in 3 parallel straight lines of which the middle one is distant from the outside ones by a distance a and b. its side = $2\sqrt{\frac{a^2+ab+b^2}{3}}$



Let CAB be the equilateral Δ and $CA=x$ its side $=x$. about the angle $DAC = \theta$

$\therefore \angle BAF = 30^\circ - \theta$

$\angle CBF = 60^\circ - \theta$

also $a+b = x\cos\theta \quad \therefore x = \frac{a+b}{\cos\theta}$

$b = x\cos(60^\circ-\theta) \therefore x = \frac{b}{\cos(60^\circ-\theta)} = \frac{b}{\cos 60^\circ\cos\theta + \sin 60^\circ\sin\theta} = \frac{b}{\frac{\cos\theta}{2} + \frac{\sqrt{3}\sin\theta}{2}}$

$a = x\sin(30^\circ-\theta) \therefore x = \frac{a}{\sin(30^\circ-\theta)} = \frac{a}{\sin 30\cos\theta - \cos 30\sin\theta} = \frac{a}{\frac{\cos\theta}{2} - \frac{\sqrt{3}\sin\theta}{2}}$

Equating the second and third equations

$(b-a)\cos\theta = (a+b)\sqrt{3}\sin\theta$

$(b^2-2ab+a^2)\cos^2\theta = 3(a^2+2ab+b^2)(1-\cos^2\theta)$

$4(a^2+ab+b^2)\cos^2\theta = a^2+2ab+b^2$

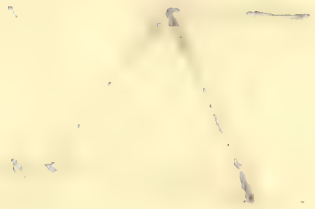
$2\sqrt{a^2+ab+b^2}\cdot\cos\theta = a+b$

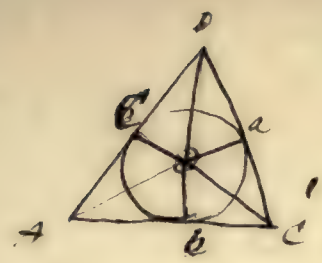
$\cos\theta = \frac{a+b}{2\sqrt{a^2+ab+b^2}}$

But from above $x = \frac{a+b}{\cos\theta}$

$x = \frac{(a+b)\,2\sqrt{a^2+ab+b^2}}{(a+b)\sqrt{3}}$

$= 2\sqrt{\frac{a^2+ab+b^2}{3}}$





$2r = \frac{b}{\sin B}$ $\therefore b = 2r \sin B$

Let B contain $60°$ $\therefore b = r$ $\frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}}$.

Let $\angle B = 60$ deg. rad of circ = 1.

Then $BC = r \cot 30 = r\sqrt{3}$ Let $AO = x$

$BO = r \operatorname{cosec} 30 = r\,2$ $OC = (a - x)$

$\frac{OC}{BC} = \pm b$. Then $AB = \sqrt{x^2 - r^2} = AC$

$\frac{BO}{r} = \cot \frac{B}{2}$ $CB\ CB = \sqrt{(a-x)^2 - r^2} = C'a$

$BO = r \cot \frac{B}{2}$ $\therefore$ perim =

$\frac{BO}{OB} = \operatorname{cosec} OB$

$BO = r \operatorname{cosec} OB$. $2\left\{r\sqrt{3} + \sqrt{x^2 - r^2} + \sqrt{(a-x)^2 - r^2}\right\}$

Then $CB \sin 60° = C'A \sin A$

$\left\{\sqrt{(a-x)^2 - r^2} + r\sqrt{3}\right\}\frac{1}{2}\sqrt{3} = \left\{\sqrt{x^2 - r^2} + \sqrt{(a-x)^2 - r^2}\right\} \sin A$

$\therefore \sin A = \dfrac{\left\{\sqrt{(a-x)^2 - r^2} + r\sqrt{3}\right\}\frac{1}{2}\sqrt{3}}{\sqrt{x^2 - r^2} + \sqrt{(a-x)^2 - r^2}} =$

Then $\frac{A}{2} \cot \frac{A}{2} = 2 \cdot \dfrac{r}{x} \cdot \dfrac{\sqrt{x^2 - r^2}}{x}$

$\therefore \dfrac{\sqrt{(a-x)^2 - r^2}\sqrt{3} + 3r}{2\sqrt{x^2 - r^2} + 2\sqrt{(a-x)^2 - r^2}} = \dfrac{2r\sqrt{x^2 - r^2}}{x^2}$

Let y = the value of x determined by elimination

then $AB = \sqrt{y^2 - r^2} + r\sqrt{3}$

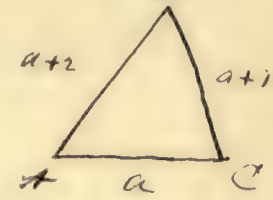
$BC' = \sqrt{(a-y)^2 - r^2} + r\sqrt{3}$

$AC' = \sqrt{(a-y)^2 - r^2} + \sqrt{y^2 - r^2}$

Trigonometry.

To determine the triangle whose sides are 3 consecutive natural numbers, and whose greatest angle is double of the least.

Let a, $a+1$, $a+2$ be the sides



Then $a \sin C = (a+2) \sin B$

$2a \sin B \cos B = (a+2) \sin B$

$$\cos B = \frac{a+2}{2a} \quad (1) \qquad \text{also } \cos B = \frac{(a+2)^2 + (a+1)^2 - a^2}{2(a+2)(a+1)} = \frac{a^2+6a+5}{2a^2+6a+4} \quad (11)$$

Equate (1) and (11) and eliminate a which $= 4$

$\therefore AC = 4$, $CB = 5$, $AB = 6$

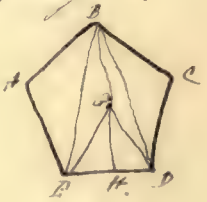
To find the angles. $\cos B = \frac{a+2}{2a} = \frac{6}{8} = \frac{3}{4}$ $\therefore \sin B = \frac{1}{4}\sqrt{7}$

$\frac{\sin A}{\sin B} = \frac{a+1}{a}$ $\therefore \sin A = \frac{5}{16}\sqrt{7}$ $\sin C = 2 \sin B \cos B = \frac{3}{8}\sqrt{7}$

$\therefore A = \angle \sin^{-1} \frac{5}{16}\sqrt{7}$ $B = \angle \sin^{-1} \frac{3}{4}$ $C \angle \sin^{-1} \frac{3}{8}\sqrt{7}$.

The alternate angles of a regular polygon being joined, is regarded to ~~find~~ compose the sum of the isosceles triangles formed with the pentagon.

Let r = radius of circumscribed circle.



Then $\angle EOH = 54°$ $\therefore GH = r \sin 54° = \frac{r}{4}(1+\sqrt{5})$

$EH = r \cos 54° = \frac{r}{4}\sqrt{10-2\sqrt{5}}$

Then area of $\triangle EOD = \frac{r^2}{16}\sqrt{40+8\sqrt{5}} = \frac{r^2}{8}\sqrt{10+2\sqrt{5}}$

Also $BH = BG + GH = r + \frac{r}{4}(1+\sqrt{5}) = \frac{r}{4}(5+\sqrt{5})$

and area of $\triangle BED = \frac{r^2}{16}\sqrt{200+40\sqrt{5}} = \frac{r^2}{8}\sqrt{50+10\sqrt{5}}$

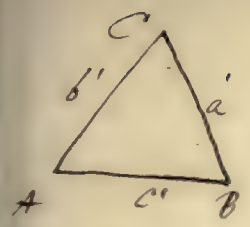
Also area of pentagon = 5 times area of triangle GED

and area of all the isosceles triangles = 5 times area of BED

Therefore the sum of the corners is to the sum of the sides as $\sqrt{10+2\sqrt{5}}$: $\sqrt{50+10\sqrt{5}}$

$\therefore 1 : \sqrt{5}$

which is the ratio required



(19)

Given $(a'+b'+c') = 2s$

$\angle A = 60°$ } to find a'

area $= m$

Then we have area $= \sqrt{s(s-a')(s-b')(s-c')} = m$ (I)

also $\tan \frac{A}{2} = \tan 30° = \sqrt{\frac{(s-b')(s-c')}{s(s-a')}} = \frac{1}{3}\sqrt{3}$ (II)

$\therefore$ by division $\frac{1}{s(s-a')} = \frac{\sqrt{3}}{3m}$ or $a' = \frac{s^2 - m\sqrt{3}}{s}$

Given $b'+c' = s$

area $= m$ } to find the sides.

$\angle A = 60°$

In the last case we have $s(s-a) = \frac{3m}{\sqrt{3}} = m\sqrt{3}$

also $\cos 30° = \sqrt{\frac{s(s-a)}{bc}} = \frac{1}{2}\sqrt{3}$

$\therefore s(s-a) = \frac{3bc}{4}$

But $s(s-a) = m\sqrt{3}$ by what has been proved

Therefore $bc = 4m\sqrt{3}$

and $(b'+c')^2 = s^2$

$4b'c' = 16m\sqrt{3}$

$b'-c' = \sqrt{3s^2 - 16m\sqrt{3}}$

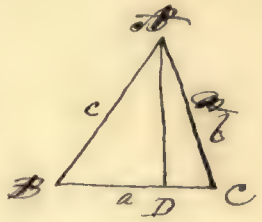
$\therefore b' = \frac{1}{2}\left\{s + \sqrt{3s^2 - 16m\sqrt{3}}\right\}$

$c' = \frac{1}{2}\left\{s - \sqrt{3s^2 - 16m\sqrt{3}}\right\}$

and from eq. $\left(\frac{a'+b'+c'}{2}\right)\left(\frac{b'+c'-a'}{2}\right) = \frac{3bc}{4}$ we may find a' —

Trigonometrical Problem.

Express the area of a triangle in terms of two of its sides and an angle opposite one of them: also, in terms of two of its angles and a side opposite one of them.



(I) Given sides a.c. angle C

$$\sin A = \frac{a}{c}\sin C$$

$$\therefore \angle B = 180^\circ - \left\{C + \sin^{-1}\frac{a}{c}\sin C\right\}$$

$$\sin B = \sin\left\{180^\circ - \left(1 + \sin^{-1}\frac{a}{c}\right)\sin C\right\}$$

area of $\Delta = \frac{BC \times AD}{2}$ if AD is perp to BC. $\because AB = c \sin B$

$$= c \sin\left\{180^\circ - \left(1 + \sin^{-1}\frac{a}{c}\right)\sin C\right\}$$

$$= \frac{ac}{2}\sin\left\{180^\circ - \left(1 + \sin^{-1}\frac{a}{c}\right)\sin C\right\} \text{ ——}$$

(II) Given $\angle B$. $\angle C$. side b.

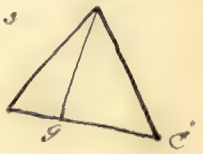
$$\sin A = \sin(180^\circ - A) = \sin(B + C)$$

$$b \sin A = a \sin B$$

$$b \sin(B+C) = a \sin B \quad \therefore a = \frac{b \sin(B+C)}{\sin B}$$

$$AD = b \sin C \quad \therefore \text{area} = \frac{b^2 \sin(B+C)\sin C}{2 \sin B}$$

To find the sides of a triangle when the base, the sides, and perp. are in continued geometric prog. Let $BC = a$, $BA = ar$, $AC = ar^2$, $AD = ar^3$



Then $\frac{AC}{BC} = \frac{\sin B}{\sin A} = \frac{ar^3}{ar} = r^2 \quad \therefore \sin B = r^2 \sin A$

Also $\sin A = \frac{BD}{AB} = \frac{a}{ar^2} = \frac{1}{r^2} \quad \therefore \sin B = \frac{r^2}{r^2} = 1$ or B is a right $\angle$ a.

Then $AC^2 = AB^2 + BC^2$ or $a^2r^6 = a^2r^4 + a^2r^2$

$$r^4 = r^2 + 1 \qquad r^4 - r^2 + \frac{1}{4} = \frac{5}{4}$$

And $\sin B = r^2 \sin A = \frac{1+\sqrt{5}}{2} = .618$ $\qquad r^2 - \frac{1}{2} = \pm\frac{1}{2}\sqrt{5}$

$$r^2 = \frac{1}{2}\left\{1 \pm \sqrt{5}\right.$$

$$\sin C = \frac{a}{ar} = \frac{1}{r} = \sqrt{\frac{2}{1+\sqrt{5}}} \qquad r = \sqrt{\frac{17+\sqrt{5}}{2}}$$

$$= \sqrt{\frac{2}{3.236}}$$

$\therefore \angle A = \angle \sin^{-1}.618$ $\angle B$ is a right angle $\angle C$ $\angle \cos^{-1}.618$

Trigonometry.

In a right angled triangle, given the radii of the circumscribed circle, and inscribed circle, to find the sides and area.

Let r = rad of inscribed circle

$(y+r)$ and $(x+r)$ = base and perpendicular of triangle

$\therefore x+y$ = hypothenuse.

Let R = rad of circumscribed circle — Then since the triangle is right angled, the hypothenuse coincides with the diameter

$\therefore x+y = 2R$. or $x = 2R-y$ (1)

Also $(x+y)^2 = (y+r)^2+(x+r)^2 \therefore 2rx = 2\{r^2+ry+rx\}$ or $x = \frac{r(r+y)}{y-r}$ (11)

Then by equating (1) and (11) a value of y will be found $= B$ suppose

$\therefore B+r$; $2R-B+r$; $2(R+r)$ are the three sides

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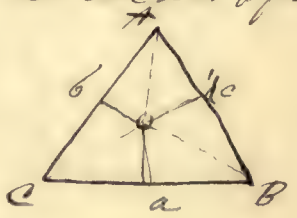
21. Given the radius of circumscribed circle the vertical angle and the ratio of the sides containing it to find the sides of the triangle



Given $\angle A = 60°$ & $b : c :: 3 : 4$

also R = radius of circumscribing $\odot$

Now $2R = \frac{a}{\sin A} = \frac{a}{\frac{1}{2}\sqrt{2}} \therefore a = R\sqrt{2}$.

~~also $A+A' = R\cos 30° = R\frac{\sqrt{3}}{2}$~~

let $c = 4x \therefore b = 3x$

Then area of triangle $= 12x^2 \sin A = \frac{12x^2\sqrt{3}}{2} = 6x^2\sqrt{3}$

$\therefore \sqrt{s(s-a)(s-b)(s-c)} = 6x^2\sqrt{3}$

$\tan\frac{A}{2} = \tan 30° = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{1}{3}\sqrt{3}$

$\frac{1}{s(s-a)} = \frac{6x^2\sqrt{3}}{\frac{1}{3}\sqrt{3}} = 18x^2$

$18x^2\left(\frac{3x+4x+R\sqrt{2}}{2}\right)\frac{3x+4x-R\sqrt{2}}{2} = 1$

$18x^2\{7x+R\sqrt{2}\}(7x-R\sqrt{2}) = 4$

$18x^2(7x^2-2R^2) = 4$

$126x^4+36R^2x^2 = 4$ from which x may be determined —

$$\frac{\sin 45^\circ - \sin 30^\circ}{\cos 45^\circ + \sin 30^\circ} = \frac{\sin 45^\circ}{\sin 45^\circ + \sin 30^\circ} - \frac{\sin 30^\circ}{\sin 45^\circ + \sin 30^\circ}$$

$$= \frac{\frac{1}{\cos 45^\circ}}{\frac{1}{\cos 45^\circ} + \frac{\sin 30^\circ}{\sin 45^\circ \cdot \cos 45^\circ}} - \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{2} + \frac{1}{2}}$$

$$= \frac{\sec 45^\circ}{\sec 45^\circ + \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{2} \cdot \frac{1}{2}\sqrt{2}}} - \frac{1}{\frac{1}{\frac{1}{2}\sqrt{2}} + 1}$$

$$= \frac{\sec 45^\circ}{\sec 45^\circ + \tan 45^\circ} - \frac{\tan 45^\circ}{\sec 45^\circ + \tan 45^\circ} = \frac{\sec 45^\circ - \tan 45^\circ}{\sec 45^\circ + \tan 45^\circ}$$

$$\frac{\sin 60^\circ}{\sin 60^\circ + \sin 30^\circ} = \frac{\frac{\sin 60^\circ}{\cos 60^\circ}}{\frac{\sin 60^\circ}{\cos 60^\circ} + \frac{\sin 30^\circ}{\sin 30^\circ}} = \frac{\tan 60^\circ}{\tan 60^\circ + 1} = \frac{\tan 60^\circ}{\tan 60^\circ + \tan 45^\circ}$$

$$\frac{\sin 30^\circ}{\sin 60^\circ + \sin 30^\circ} = \frac{\frac{\sin 30^\circ}{\sin 30^\circ}}{\frac{\sin 60^\circ}{\cos 60^\circ} + \frac{\sin 30^\circ}{\sin 30^\circ}} \quad \frac{1}{\tan 60^\circ + 1} = \frac{\tan 45^\circ}{\tan 60^\circ + \tan 45^\circ}$$

Ex 30

$$(s^2 - s^2 s^2)(c^2 - c^2 c^2) = (s^2 - s^2 s^2)(c^2 - c^2 c^2)$$

$$s^2(1 - s^2)\{c^2(1 - c^2)\} = s^2(1 - s^2)\{c^2(1 - c^2)\}$$

$$s^2 c^2 c^2 s^2 = s^2 c^2 c^2 s^2$$

$$\because (1 - s^2) = c^2$$
$$(1 - c^2) = s^2$$
$$(1 - s^2) = c^2$$
$$(1 - c^2) = s^2$$

Ex 31

$$\frac{s^2(s^2 - 1)}{c^2(s^2 - 1)} = \frac{s^2\left(\frac{1}{\cos^2} - 1\right)}{s^2\left(\frac{1}{\cos^2} - 1\right)} = \frac{\frac{1}{\cos^2}\left(\frac{1 - \cos^2}{\cos^2}\right)}{\frac{1}{\cos^2}\left(\frac{1 - \cos^2}{\cos^2}\right)} = \frac{\frac{\sin^2 + \cos^2}{\cos^2}\left(\frac{\sin^2}{\cos^2}\right)}{\frac{\sin^2 + \cos^2}{\cos^2}\left(\frac{\sin^2}{\cos^2}\right)} = \frac{\left(\frac{\sin^2}{\cos^2} + 1\right)t^2}{\left(\frac{\sin^2}{\cos^2} + 1\right)T^2} = \frac{t^2(T$$

$$\frac{T}{t} = \frac{s}{s}$$

$$\frac{\frac{s}{c}}{\frac{s}{c}} = \frac{s}{s}$$

$$\frac{\frac{1}{c}}{\frac{1}{c}} = 1$$

$$\frac{c}{c} = 1$$

$$c = C$$

Ex 32)

Problem 1 Let A and A' represent the two angles — their arcs
a and a' — their radii
r and r' —

Then $a : a' :: \frac{1}{r} : \frac{1}{r'}$ or $a = \frac{a'r'}{r}(1) = a'\left(\frac{r'}{r}\right)$

But $A = \frac{a}{r}$ and $A' = \frac{a'}{r'}$

$\therefore A = A'$

$\frac{a}{r} = \frac{a'}{r'}$

$ar' = a'r$

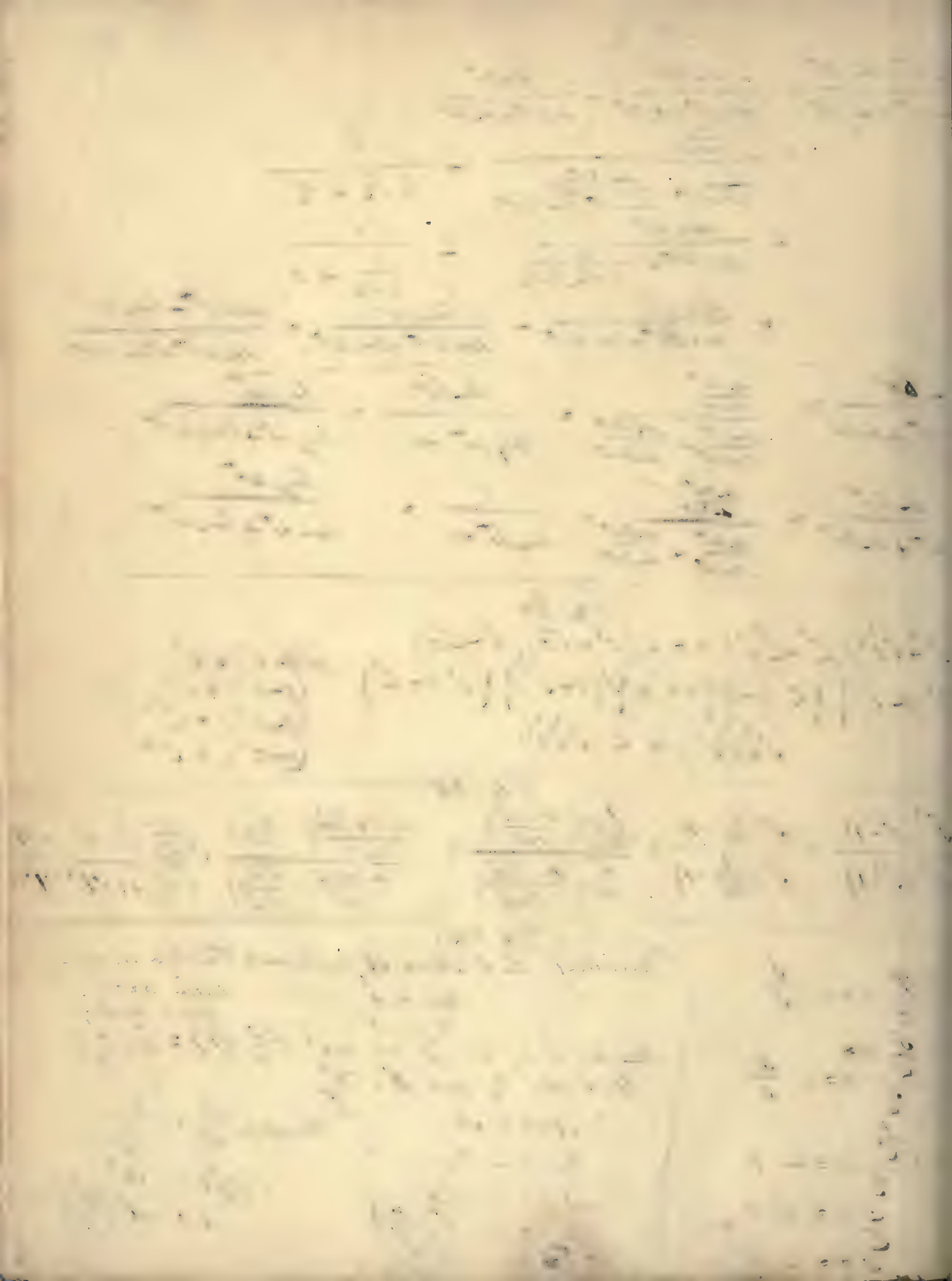
$a = a'\left(\frac{r}{r'}\right)$

$a'\left(\frac{r'}{r}\right) = a'\left(\frac{r}{r'}\right)\left(\frac{r}{r'}\right)^2$

Therefore $\frac{A}{A'} = \frac{1}{\left(\frac{r}{r'}\right)^2}$

$\cdot\left(\frac{r}{r'}\right)^2 A = A'$

$A = A'\left(\frac{r'}{r}\right)^2$



Ex 26

Prove $(\sin 30^\circ + \cos 30^\circ)(\sin 60^\circ - \cos 60^\circ) = \sin 30^\circ$

Let $30^\circ = A$
$\therefore 60^\circ = 2A$
$\sin 30^\circ = \sin A$
$\cos 30^\circ = \cos A$
$\sin 60^\circ = \sin 2A$
$\cos 60^\circ = \cos 2A$

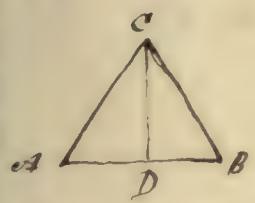
$(\sin(30^\circ + \cos 30^\circ)(\sin 60^\circ - \cos 60^\circ)$
$= (\sin A + \cos A)(\sin 2A - \cos 2A)$
$= \{\sin A + \sqrt{1-\sin^2 A}\}\{2\sin A\sqrt{1-\sin^2 A} - (1-2\sin^2 A)\}$
$= (4\sin^2 A - 1)\sqrt{1-\sin^2 A} + \sin A$ by multiplication
$= \sin A$ for $(4\sin^2 A - 1) = (4(\tfrac{1}{2})^2 - 1) = 0$

27

Prove $2\sin 30^\circ \cdot \cos 30^\circ = \sin 60^\circ$

Let $30^\circ = A$
$60^\circ = 2A$.

$\sin(A+B) = \sin A \cos B + \cos A \sin B$
Let $A = B$ then $\sin 2A = 2\sin A \cos A$. (I)



Let ABC be an equilateral triangle then $\tan CAB = \tan 60^\circ = \frac{CD}{AD} = \frac{AD\sqrt{3}}{AD} = \sqrt{3}$
$\sin CAD = \sin 60^\circ = \frac{CD}{AC}$ i.e. $\sqrt{\frac{3AD^2}{4AD^2}} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$
But $\tan 60^\circ = \sqrt{3}$ therefore $\tan 60^\circ = 2\sin 60^\circ$ (II)

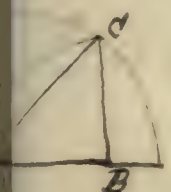
Also $\tan A+B = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ if $A = B$ and $\tan 2B = \frac{2\tan B}{1-\tan^2 B}$
Now let $B = 30^\circ$ and we have $\tan 60^\circ = \frac{2\tan 30^\circ}{1-\tan^2 30^\circ} = \frac{2\tan 30^\circ}{1-\frac{1}{3}} = 3\tan 30^\circ$ (III)

Let CAB be a semicircle, and $AB = \text{chd } 120^\circ \therefore CA = \text{chd } 60^\circ = 1$
Then $\angle CAB$ in a semicircle is a right angle therefore $CB^2 = CA^2 + AB^2$
or $4 = 1 + AB^2 \therefore AB = \sqrt{3} = \tan 60^\circ$ proved above (IV)

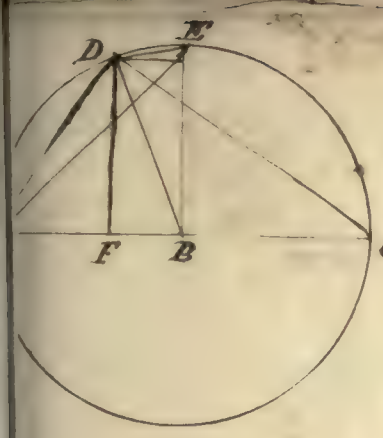
$\tan 60^\circ = \frac{2\tan 30^\circ}{1-\tan^2 30^\circ} \therefore \frac{2}{3}\tan 60^\circ = 2\tan 30^\circ$ (V)

$\operatorname{cosec} 60^\circ = \frac{1}{\sin 60^\circ} = \frac{\cos 30^\circ}{\sin 60^\circ \cos 30^\circ} = \frac{1}{\sqrt{3}\cos 30^\circ} = \frac{\frac{2}{3}(\frac{1}{2}\sqrt{3})}{\cos 60^\circ} = \frac{2}{3}\left(\frac{\sin 60^\circ}{\cos 60^\circ}\right) = \frac{2}{3}\tan 60^\circ$

28



Let ABC be an isosceles triangle having $\angle CBA$ a right $\angle \therefore \angle CAB = 45^\circ$.
$\sin CAB = \frac{CB}{CA} = \frac{CB}{CB\sqrt{2}} = \frac{1}{\sqrt{2}}$ and $\cos 45^\circ = \sin(90^\circ - 45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}$
$\therefore \sin 45^\circ + \cos 45^\circ = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} = \sqrt{2}$
Also $\angle CBA = 90^\circ$ the chord of which $= CA$
But $CA^2 = CB^2 + BA^2 = 2BA^2$ where of $BA = r = 1$
$\therefore CA^2 = 2$
$CA = \sqrt{2} =$
$\sin 45^\circ + \cos 45^\circ$ as above



$$Chd\,A\,chd(\pi - A) =$$

$$c\left\{chd\frac{\pi}{2} + chd\left(\frac{\pi}{2} - A\right)\right\}\left\{chd\frac{\pi}{2} - chd\left(\frac{\pi}{2} - A\right)\right\}$$

Let DBA = ang A.

Then from the triangle DFA right angled at F

$DA^2 = DF^2 + AF^2$ or $Chd^2 A = sin^2 A + ver^2 A$

Also $chd^2\left(\frac{\pi}{2} - A\right) = chd^2 DBE = (1 - ver A)^2 + (1 - sin A)^2$

$Chd^2(\pi - A) = chd^2 DBC = sin^2 A + (2 - ver A)^2$

} assuming radius of circle = 1

$$Chd\,\frac{\pi}{2} = AE = \sqrt{2}$$

Then $Chd\,A\,chd(\pi - A) = \sqrt{sin^2 A + ver^2 A}\,\sqrt{sin^2 A + (2 - ver A)^2}$

$$= \sqrt{\{sin^2 A + ver^2 A\}\{sin^2 A + 4 - 4 ver A + ver^2 A\}}$$

$$= \sqrt{(3)} = \sqrt{\left(sin^4 A + ver^4 A + 4 sin^2 A + 4 ver^2 A - 4 ver^3 A - 4 ver A\, sin^2 A + 2 sin^2 A\, ver^2 A\right)}$$

Also from above $Chd\frac{\pi}{2} + Chd\left(\frac{\pi}{2} - A\right) = \sqrt{2} + \sqrt{(1 - ver A)^2 + (1 - sin A)^2}$

$$Chd\,\frac{\pi}{2} - Chd\left(\frac{\pi}{2} - A\right) = \sqrt{2} - \sqrt{(1 - ver A)^2 - (1 - sin A)^2}$$

$$\left\{Chd\frac{\pi}{2} + Chd\left(\frac{\pi}{2} - A\right)\right\}\left\{Chd\frac{\pi}{2} - Chd\left(\frac{\pi}{2} - A\right)\right\} = 2 ver A + 2 sin A - sin^2 A - ver^2 A \quad (\gamma)$$

Moreover in the similar triangles CDA, DAE

$AC : AD :: AD : AE$

$AC \times AE = AD^2$ or $2 ver A = Chd^2 A$

$Chd^2 A = ver^2 A + sin^2 A \therefore 2 ver A = ver^2 A + sin^2 A$

$8 ver A = 4 ver^2 A + 4 sin^2 A$

$8 ver A sin A = 4 ver^2 A sin A + 4 sin^3 A$

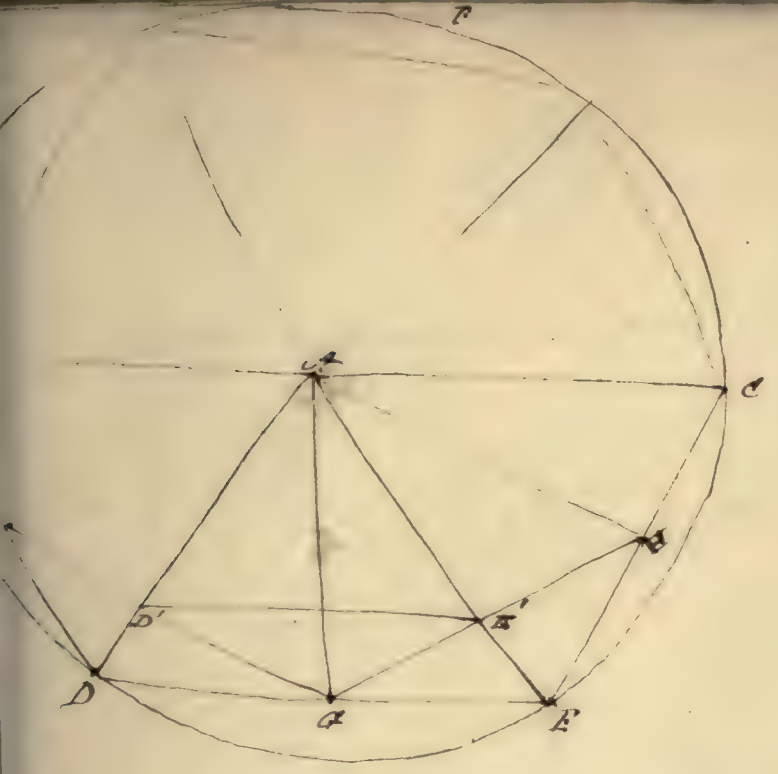
} By multiplication

$$8 ver A \cdot sin A - 4 ver^2 A\, sin A - 4 sin^3 A = 0 \quad (\delta)$$

Addition of β & δ

$$A \cdot chd(\pi - A) = \sqrt{\left(A^4 + ver^4 A + 4 sin^2 A + 4 ver^2 A - 4 ver^3 A + 4 sin^2 A - 4 ver A\, sin^2 A - 4 ver^2 A\, sin^2 A + 8 ver A\, sin^2 A + 2 sin^2 A\, ver^2 A\right)}$$

$2 ver A + 2 sin A - sin^2 A - ver^2 A = (\gamma)$ as required.



Let BDECF repr the circle whose BC is the chord and cutn A, also let BD, DE, EC be the sides of an hexagon inscribed in the ○ ∴ DE = AD or AE & bisect these sides in F, G, H and join FG, GH also let FG and GH be agn bisected in D', E' and join D'E'.

From A draw AG, which by Euc will be at ∠ to DE

Let AD = r the radius of the circle. Now as the ∠ADE is equiangular, each of its ∠s will be 60° also AG = r sin ADG

$= r \sin 60° = r\left(\frac{1}{2}\sqrt{3}\right)$

But AG = GH for GH is the side of an equiangular & equilateral hexagon inscribed in a circle whose rad = AG ∴ $GH = r\left(\frac{1}{2}\sqrt{3}\right)$

Also $AE' = AG \sin AGE' = r\left(\frac{1}{2}\sqrt{3}\right)\sin 60° = r\left(\frac{1}{2}\sqrt{3}\right)^2$

In the same manner the side of the hexagon produced by the next bisection might be shown to be equal to $r\left(\frac{1}{2}\sqrt{3}\right)^3$

∴ the sum of the perimeters of the hexagons would be 6 times the sum of a geometric series ad inf, whose first term = r and the ratio = $\frac{1}{2}\sqrt{3}$ Let r = 1st term and $\left(\frac{1}{2}\sqrt{3}\right) = s$

$$\Sigma' = \frac{r}{1-s} = \frac{r}{1-\frac{1}{2}\sqrt{3}} = \frac{2r}{2-\sqrt{3}}$$

and $\left(\frac{2r}{2-\sqrt{3}}\right)6 = \frac{12r}{2-\sqrt{3}}$ ~~$= \frac{24r + 12r\sqrt{3}}{} = 12r(2+\sqrt{3})$~~

∴ the perimeter of the sides = ~~$12r(2+\sqrt{3})$~~ $\frac{12r}{2-\sqrt{3}}$

II. To find the areas of the hexagons.

The area of the larger hexagon = 6 area of Δ ADE = 3 AD or DE × AG

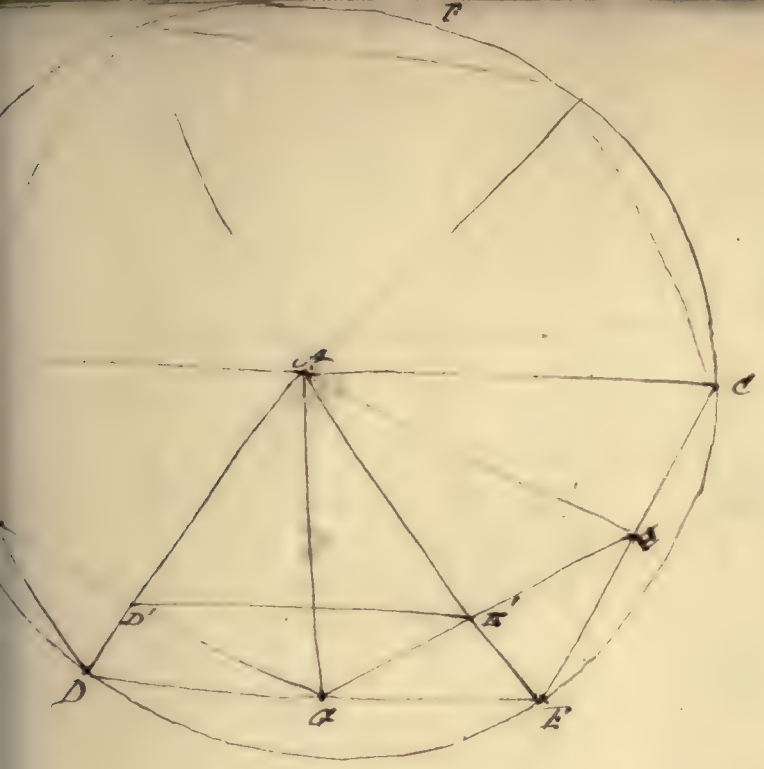
$= 3r\left(r\,\frac{1}{2}\sqrt{3}\right) = 3r^2\,\frac{1}{2}\sqrt{3}$.

The area of the next smaller hex = 6 area of Δ AGH = 3 AG × AE'

$= 3\left(r\,\frac{1}{2}\sqrt{3}\right)\left(r\,\frac{1}{2}\sqrt{3}\right)^2 = 3r^2\left(\frac{1}{2}\sqrt{3}\right)^3$

& the area of all the hexagons $= 3r^2\left\{\left(\frac{1}{2}\sqrt{3}\right) + \left(\frac{1}{2}\sqrt{3}\right)^3 + \left(\frac{1}{2}\sqrt{3}\right)^5 + \text{&c. ad inf}\right\}$

$= 3r^2(2\sqrt{3}) = r^2(6\sqrt{3})$



Let BDECE ~~of~~ circle where BC is the diameter and centre A, also let BD, DE, EC be three sides of an hexagon inscribed in the ⊙ ∴ DE = AD or AE & bisect these sides in F, G, H & join FG, GH also let FG & GH be again bisected in D', E' and join D'E'

From A draw AG, which by Euc will be at ∠ to DE

Let AD = r the radius of the circle. Now as the ∠ ADE is equiangular, each of its ∠s is only 60°

also AG = r sin ADG

= r sin 60° = r $(\frac{1}{2}\sqrt{3})$

But AG = GH for GH is the side of an equiangular & equilateral hexagon inscribed in a circle whose rad = AG ∴ GH = r $(\frac{1}{2}\sqrt{3})$

also AE' = AG sin AGE' = r $(\frac{1}{2}\sqrt{3})$ sin 60° = r $(\frac{1}{2}\sqrt{3})^2$

In the same manner the side of the hexagon produced by the next bisection might be shown to be equal to r $(\frac{1}{2}\sqrt{3})^3$

∴ the sum of the perimeters of the hexagons would be 6 times the sum of a geometric series ad inf, whose first term = r and the ratio = $\frac{1}{2}\sqrt{3}$ Let r = 1st term and $(\frac{1}{2}\sqrt{3})$ = s

$$\Sigma' = \frac{r}{1-s} = \frac{r}{1-\frac{1}{2}\sqrt{3}} = \frac{2r}{2-\sqrt{3}}$$

and $\left(\frac{2r}{2-\sqrt{3}}\right) 6 = \frac{12r}{2-\sqrt{3}}$ ~~= 24r + 12r√3 = 12r(2 + √3)~~

∴ the perimeter of the sides = ~~12r(2+√3)~~ $\frac{12r}{2-\sqrt{3}}$

II. To find the areas of the hexagons.

The area of the largest hexagon = 6 area of Δ ADE = 3 AD or DE × AG

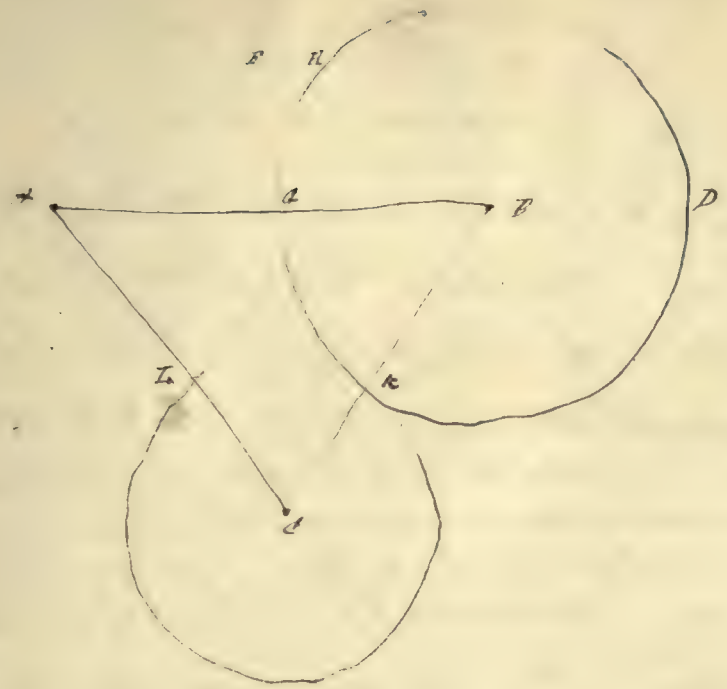
= 3r (r $\frac{1}{2}\sqrt{3}$) = 3r² $(\frac{1}{2}\sqrt{3})$.

The area of the next smaller hex = 6 area of Δ AGH = 3 AH × AE'

= 3 (r $\frac{1}{2}\sqrt{3}$) (r $\frac{1}{2}\sqrt{3}$)² = 3r² $(\frac{1}{2}\sqrt{3})^3$

& the areas of all the hexagons = 3r² $\{(\frac{1}{2}\sqrt{3}) + (\frac{1}{2}\sqrt{3})^3 + (\frac{1}{2}\sqrt{3})^5$ &c - ad inf &c $\}$

= 3r² $(2\sqrt{3})$ = r² $(6\sqrt{3})$



Let A, B, C, be the centres of the three circles.

Then because the circles, BFG, GHD, touch one another, and AB a line joining their centres has been drawn, it passes through the point of contact G. Similarly it may be shewn that the other lines BC, AC respectively pass through the points of contact K, I₂.

Now let AG or $AI_2 = r$

BG or $BK = r'$

CI₂ or $CK = r''$

$\therefore AB = r + r' = c$

$AC = r + r'' = b$

$CB = r' + r'' = a$

Then $\sin A = \frac{2}{(r+r')(r+r'')} \sqrt{s\{s-(r'+r'')\}\{s-(r+r'')\}\{s-(r+r')\}}$

where $s = \frac{1}{2}$ the perimeter of the sides

$$\sin B = \frac{2}{(r'+r'')(r+r')} \sqrt{s\{s-(r'+r'')\}\{s-(r+r'')\}\{s-(r+r')\}}$$

$C = 180^\circ - (A+B)$ which gives the angles.

Then area of $\Delta ABC = \sqrt{s\{s-(r'+r'')\}\{s-(r+r'')\}\{s-(r+r')\}}$

area of sector $AGI_2 = \frac{360^\circ}{\angle A}(r^2\pi) = \frac{\angle A}{360^\circ}\cdot(r^2\pi)$

" " $BGK = \frac{360^\circ}{\angle B}(r'^2\pi) = \frac{\angle B}{360^\circ}\cdot(r'^2\pi)$

" " $GI_2K = \frac{360^\circ}{\angle C}(r''^2\pi) = \frac{\angle C}{360^\circ}\cdot(r''^2\pi)$

$\therefore$ area of the three sectors $= \frac{\pi}{360^\circ}\{Ar^2 + Br'^2 + Cr''^2\}$

& the difference between these areas gives the area of the inclosed space. GKI_2.

Prob 24. $\sin A(\sin A - \cos A) = m \therefore \sin^2 A - \sin A\sqrt{1-\sin^2 A} = m$

$2\sin^4 A - (2m+1)\sin A = -m^2$ a quadratic which would give $\sin A$

Prob 25. $m \operatorname{vers} A \pm n \operatorname{vers}(\pi - A) = p$ $\because \operatorname{vers}(\pi - A) = 2 - \operatorname{vers} A$.

$\therefore m \operatorname{vers} A \pm n(2 - \operatorname{vers} A) = p \therefore$ from which value of $\operatorname{vers} A$ may be found

---

Prob 26. $\tan A + \cot A = 4 \therefore \tan A + \frac{1}{\tan A} = 4 \therefore \tan^2 A - 4\tan A = -1$.

---

27. $m \tan A + n \cot A = p \sec A \therefore m\frac{\sin A}{\cos A} + n\frac{\cos A}{\sin A} = p\frac{1}{\cos A}$

$m(\sin A + n\left(\frac{\cos^2 A}{\sin A}\right) = p \therefore m\sin^2 A + n\cos^2 A = p\sin A$

$\therefore m\sin^2 A + n(1 - \sin^2 A) = p\sin A$ or $(m-n)\sin^2 A - p\sin A = -n$

a quadratic eqn from which $\sin A$ may be found $\therefore$ Then $\operatorname{cosec} A = \frac{1}{\sin A}$

---

28) $\sin A + \sin B = m \therefore$ also $\sin^2 A + \sin^2 B = 1 \therefore$ by divn $\sin A - \sin B = \frac{1}{m}$

whence by addit & subtract $\sin A$ & $\sin B$ may be determined

---

29 $\sin A \sec B = m \therefore \sec B = \frac{m}{\sin A} \therefore$ also $\cos A \operatorname{cosec} B = \frac{\cos A}{n} = n$

$\therefore \sec B = \frac{\cos A}{n} = \frac{m}{\sin A} \therefore \frac{\cos^2 A}{n^2} = \frac{m^2}{\sin^2 A}$ or $\frac{1-\sin^2 A}{n^2} = \frac{m^2}{\sin^2 A}$

or $\sin^4 A - \sin^2 A = (mn)^2$ or $\frac{\cos^2 A}{n^2} = \frac{m^2}{1-\cos^2 A} \therefore \cos^4 A - \cos^2 A = -m^2n^2$

from which $\cos A$ may be determined

Then $\sec A = \frac{1}{\cos A}$ and $\sec B = \frac{\cos A}{n}$

$\operatorname{Chd}^2(90+\alpha)\operatorname{chd}^2(90-\alpha) = 4r^2\{(\operatorname{vers} 90 + \alpha).\operatorname{vers}(90-\alpha)\}$

$\operatorname{Chd}^2(90+\alpha)\operatorname{chd}^2(90-\alpha) = \{\sin^2(90+\alpha) + \operatorname{vers}^2(90+\alpha)\}\{\sin^2(90+\alpha) + \operatorname{vers}^2(90-\alpha)\}$

$\sin^4(90+\alpha) + \sin^2(90+\alpha)\operatorname{vers}^2(90-\alpha) + \operatorname{vers}^2(90+\alpha)\sin^2(90+\alpha) + \operatorname{vers}^2(90+\alpha).\operatorname{vers}^2(90-\alpha)$

$= 2\sin^4(90+\alpha) + \sin^2(90+\alpha)\operatorname{vers}^2(90-\alpha) + \operatorname{vers}^2(90+\alpha)\sin^2(90+\alpha)$

$= \{\operatorname{vers}^2(90-\alpha) + \operatorname{vers}^2(90+\alpha) + 2\sin^2(90+\alpha)\sin^2(90+\alpha)\}\sin^2(90+\alpha)$

$= \{\operatorname{vers}^2(90-\alpha) + 2\operatorname{vers}(90-\alpha)\operatorname{vers}(90+\alpha) + \operatorname{vers}^2(90+\alpha)\}\sin^2(90+\alpha)$

$= \{\operatorname{vers}^2(90-\alpha) + \operatorname{vers}(90+\alpha)\}^2 \sin^2(90+\alpha)$

$= D^2\sin^2(90+\alpha) = 4r^2.\operatorname{vers}(90+\alpha)\operatorname{vers}(90-\alpha)$

Prob 17. $2\sin A = \tan A \therefore 2\sin A = \frac{\sin A}{\cos A} \therefore 2 = \frac{1}{\cos A}$ or $\cos A = \frac{1}{2}$

also $2\sin A \cos A = \sin A$ and $\sin^2 A + \cos^2 A = 1 \therefore \sin A + \cos A = \sqrt{1+\sin A}$

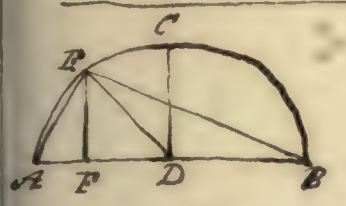
But $\cos A = \frac{1}{2} \therefore \sin A + \frac{1}{2} = \sqrt{1+\sin A}$ or $\sin A = \frac{1}{2}\sqrt{3}$

Prob 18. $\operatorname{vers}\left(\frac{\pi}{2} - A\right) = v$ But $\operatorname{vers}\left(\frac{\pi}{2} - A\right) = 1 - \cos\left(\frac{\pi}{2} - A\right)$

$\therefore 1 - \cos\left(\frac{\pi}{2} - A\right) = v \therefore \cos\left(\frac{\pi}{2} - A\right) = 1 - v$

Also $\sin^2\left(\frac{\pi}{2} - A\right) + \cos^2\left(\frac{\pi}{2} - A\right) = 1 \therefore \sin^2\left(\frac{\pi}{2} - A\right) = 1 - \cos^2\left(\frac{\pi}{2} - A\right) =$

$1 - (1-v)^2 = 2v - v^2$ Therefore $\sin\left(\frac{\pi}{2} - A\right) = \sqrt{2v - v^2}$



Let ACB be a semicircle whereof AB is the diameter, and let EDB $= (\pi - A)$ EB its chord $= k$, and since rad $= r$ the whole diameter $AB = 2r$. From E F at right angles to AD join EF $= \sin(\pi - A)$. Now since $\angle AEB$ is a right angle for it is in the semicircle AEB we have $AB^2 = AE^2 + EB^2$ or $4r^2 = AE^2 + k^2$

Therefore $AE^2 = 4r^2 - k^2$ or Rad $A = \sqrt{4r^2 - k^2}$

Also in the similar triangles AEB, EAF, $AB : EB :: EA : EF$

or $2r : k :: \sqrt{4r^2 - k^2} : \sin(\pi - A)$ from which $\sin(\pi - A) = \frac{k}{2r}\sqrt{4r^2 - k^2}$

and since $\cos^2(\pi - A) = 1 - \sin^2(\pi - A)$ we have $\cos \pi - A = \frac{2r - k^2}{2r}$

From the sine & cos all the other trigonometrical ratios may be discovered

---

Prob 20. $\sin A \cos A = m \therefore \sin A \sqrt{1 - \sin^2 A} = m$ or $\sqrt{1 - \sin^2 A} = \frac{m}{\sin A}$

from which $\sin A = \sqrt{\frac{1}{2}\left\{(1 - 4m^2)^{\frac{1}{2}} \pm 1\right\}}$

or by another method $\sin A \cos A = m \therefore 2\sin A \cos A = 2m$

Also $\sin^2 A + \cos^2 A = 1$ Therefore by addition and subtraction and extracting the square roots

$\sin A + \sin B = \sin A + \cos A = \sqrt{1 + 2m}$

$\sin A - \cos A = \sqrt{1 - 2m}$

whence by addition $\sin A = \frac{1}{2}\left\{\sqrt{1+2m} + \sqrt{1-2m}\right\}$

and by subt. $\cos A = \frac{1}{2}\left\{\sqrt{1+2m} - \sqrt{1-2m}\right\}$

Prob 21. $m \sin A = n \cos^2 A = n(1 - \sin^2 A) \therefore n \sin^2 A + m \sin A = n$

$\sin^2 A + \frac{m}{n}\sin A = 1$ from which $\sin A$ may be found by solving a quadratic then $\cos A$ may be found from the formula $\cos A = \sqrt{1 - \sin^2 A}$ and from these the other trigonometrical ratios may be discovered

---

Prob 22. $m \sin A + n \cos A = p \therefore m \sin A + n\sqrt{1 - \sin^2 A} = p$ or $m \sin A - p = n\sqrt{1 - p^2}$

which by solving a quadratic would give the value of $\sin A$.

Prob 23. $m \operatorname{ch} A + n \cos A = p \therefore \operatorname{ch} A = \frac{p^2 - 2pn\cos A + n^2\cos^2 A}{m^2}$

But $\sin^2 A \operatorname{ch} A = (1 - \cos A)^2 + \sin^2 A = 2 - 2\cos A$

$\therefore 2 - 2\cos A = \frac{p^2 - 2pn\cos A + n^2\cos^2 A}{m^2} \therefore n\cos^2 A + 2(m - pn)\cos A = m^2 - p^2$

a quadratic for $\cos A$ one solution of which would give the value of $\cos A$ which by substitution in the equation $\operatorname{ch} A = p - n\cos A$ would give its value

Prob 2

angle varies as $\frac{arc}{rad}$ Now employing c to represent the length of the arc Therefore $\frac{a}{r}$ = angle given : But the required angle is to be m times as great as this = $\frac{ma}{r}$ And it is to be subtended by a radius = $\frac{r}{n}$ ∴ employing a' to represent the length of this arc we have $\frac{a'}{\frac{r}{n}} = \frac{ma}{r}$ or $a'n = ma$ ∴ $a' = (\frac{m}{n})a$

Prob 3

Since by Euc all the interior angles of a regular polygon are equal to twice as many right angles as the figure has sides less by 4 rt ∠s we have each of the angles of a polygon of $2n$ sides = $\frac{4(n-1)\frac{\pi}{2}}{2n}$ n sides = $\frac{2(n-2)\frac{\pi}{2}}{n}$

Now by the question $\frac{2(n-1)}{n}\frac{\pi}{2} : \frac{2(n-2)}{n} :: 3 : 4$ ∴ $n = 5$ and $2n = 10$

Prob 4

Let n = number of sides in the one polygon, consequently $n+1$ = no. of sides in the other : Then each of the latter called the one of the angles of the first would contain $\left\{\frac{2(n-2)90^\circ}{n}\right\}^\circ$ and one angle of the latter would contain $\left\{\frac{2(n+1)90^\circ}{n}\right\}^\circ$

Now by the question $\frac{2(n-2)90}{n} + 4 = \frac{2(n-1)90}{n}$ ∴ $n = 45$ & $n+1 = 46$

Prob 5

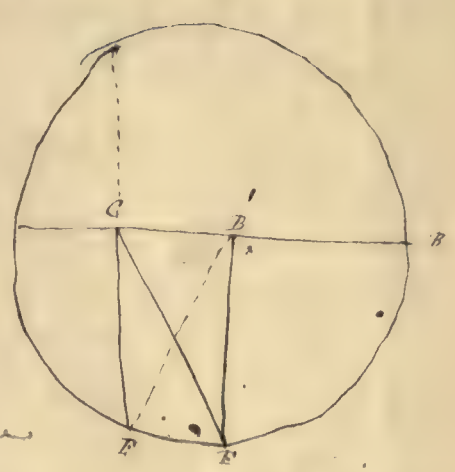
Let n = no ∴ $\left\{120 + (n-1)5\right\}\frac{n}{2}$ = the number of degrees in all the interior angles

Also by Euc $(2n-4)90$ = ---

∴ $\left\{120 + (n-1)5\right\}\frac{n}{2} = (2n-4)90$ a quadratic equation

Geometrical Problems

If two right lines be drawn from any point in the diameter of a semicircle, the one perpendicular to the diameter and the other point to the middle point of the circumference, the sum of their squares will be equal to twice the squares of the radius of the circle



Let C be the point in the diameter AB then because CE is drawn to the circumference perpendicular to the diameter it is a mean proportional between the segments of the diameter

$\therefore CE^2 = AC \times CB.$

To each of these equals add $CE^2 = BF^2 + CB^2$ if B is the centre

$\therefore CE^2 + CE^2 = BF^2 + CB^2 + AC \times CB.$

~~But the squares of CE and CB = BE^2 = BE^2~~

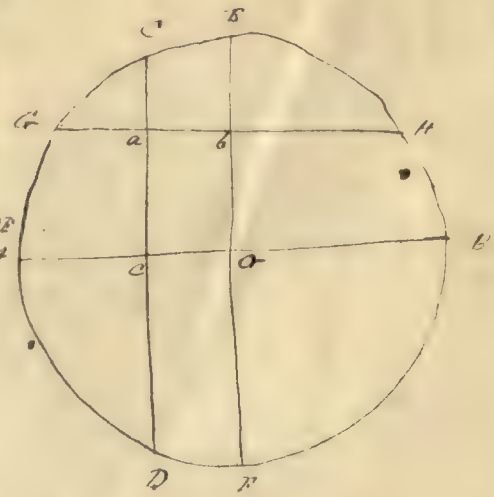
$\therefore 2BE^2 = $ ~~B~~

And because AB is divided into two equal parts in B and into two unequal parts in C; $BC^2 + AC \times CB = AB^2$ by Euc $= BE^2$

$\therefore CE^2 + CE^2 = 2BE^2$

which was to be proved.

Prop 13. Let AB, CD be the two straight lines and E AB diameter.



the arc GB = arc EH.
arc GE = arc HF
arc CE = arc DF

$\therefore$ arc GE − arc CE + arc HF + arc DF
= arc GE + arc HF
= semi-circumference.

then CD, GH are bisected in a and c

and $(Ga)^2 + (aH)^2 = 2(Gb^2 + ba^2) = 2(Gb \times bH) + 2ba^2$
$= 2(Eb \times bF) + 2ac^2$

$Ca^2 + aD^2 = 2(Cc^2 + ac^2) = 2(Cc \times cD) + 2ac^2$
$= 2(AC \times CB) + 2bo^2$

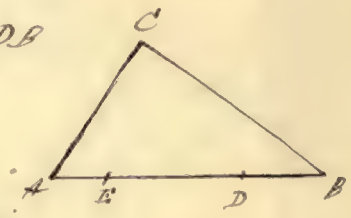
$\therefore Ga^2 + aH^2 + Ca^2 + aD^2 = 2Eb \times bF + 2bo^2 + 2\{AC \times CB + Co^2\}$

$= 2Eo^2 + 2Ao^2 = 4Ao^2 = Ao^2 + oB^2 + 2Ao \times oB = AB^2$

Geometrical Problems.

If parts be cut off from the ends of the hypothenuse of a right angled triangle, equal to the adjacent sides, the square of the middle part thus formed will be equal to twice the rectangle of the four extremes.

Let $AD = AC$ and $BE = BC$, DE^2 shall equal $2AE \times DB$



$AB^2 = AC^2 + CB^2 = AD^2 + EB^2$

$AD^2 + DB^2 + 2AD \times DB = AD^2 + EB^2$

$DB^2 + 2AD \times DB = EB^2 = ED^2 + DB^2 + 2ED \times DB$

$2AE \times DB + 2ED \cdot DB = ED^2 + 2ED \times DB$

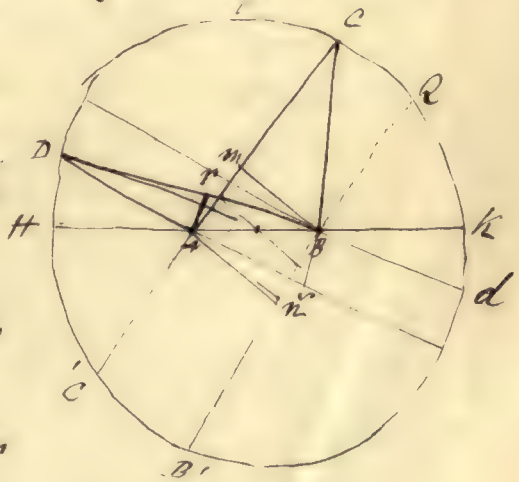
$2AE \times DB = DB^2$ ——— Q. E. D.

If right lines be drawn from any two points in the diameter of a circle equidistant from the centre, to meet two by two in the circumference, the sum of the squares of every two of them will be equal to each other.

$AC^2 + BC^2 = AD^2 + BD^2$ —

This is because the circle is symmetrically constructed with respect to its diameter.

$BB' = AC$, $BQ = AC'$. Draw Bm. Am perpn to CC' and An, Br perpendic to Dd



Then by Euc BII. $AB^2 = AC^2 + BC^2 - 2AC \cdot Cm$

$AB^2 = AD^2 + DB^2 - 2BD \cdot Br$

But $Cm = B'n = C'A$ $\therefore$ $AB^2 = AC^2 + BC^2 - 2AC \cdot AC' = AC^2 + BC^2 - 2HA \times AK$

Similarly $Dr \times DB = DB \times Bd = HB \times BK = HA \cdot AK$

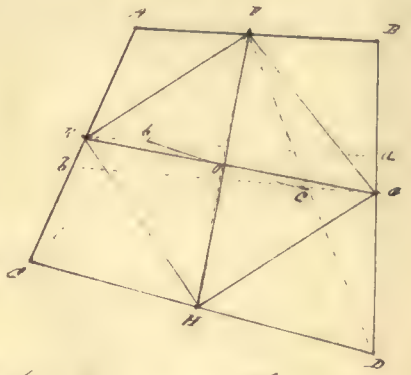
$\therefore$ $AC^2 + BC^2 - 2HA \times AK = AD^2 + BD^2 - 2HA \times AK$

or $AC^2 + BC^2 = AD^2 + BD^2$

Q. E. D.

Geometrical Prob

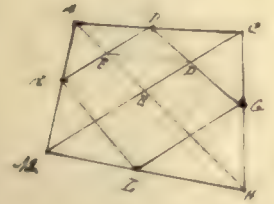
If the four sides of any trapezium be bisected, the figure formed by joining the points of bisection is a parallelogram, which is equal to half the trapezium.



Let ABCD be the trapezium, and E, F, G, H the points of bisection — join EG, FH, they shall intersect each other in O.

Through E, G draw Ea, Gb, parallel to AB and join EC, ED, and all the points bc. and because AC is bisected in E and EB is parallel to AE, CE is bisected in b — similarly ED is bisected in C and because Fb : bc :: Fc : cD, bc is parallel to CD, and therefore EO : OH :: Eb : or EO = OH. And because CD is bisected in H and bc is parallel to CD, bc is bisected in O or bO = Oc. also since Cb is the half of CE Fb is the half of AE, and for the same reason cG is the half of EB or AE. Therefore Eb = cG. And because Ea is parallel to bG and bC meets them the angle EbO is equal the angle OcG Join Ea, &c. Then the triangles EbO, OcG have the sides Eb, bO equal to the sides Gc, cO. each to each and the angle EbO has been shewn to be equal to the angle OcG therefore the base EO is equal to the base OG, and the angle bOE equal to the angle cOG. And because bO makes with bc the angles EbO, GOc these are together equal to two right angles; but the angle GOc is equal to the angle EOb; therefore the angles GOb, EOb are two right angles or EO, OG are in one straight line, and EO has been shewn to be equal to OG, and EO to OH

Join EF, EG, GH, HE. And because EO is equal to OG, and OF to OH, the angle EOF to the angle HOG, the base EF is equal to the base HG. Similarly EG is equal to EH. Therefore the two EF, FG are equal to the two GH, HE each to each and the base EG is common therefore the angles EFG, EHG are equal, and for the same reason the angles EFH, EGH are equal — and since the four angles EFG, EGH, GHE, HEF are together equal to 4 right angles, the two angles EFG, EGH are equal to two right angles, and therefore EF is parallel to GH, and similarly EH is parallel to FG



Because CB is the half of BA and CG of GH. BC is parallel to AH. and KL is parallel to AH, for the same reason ∴ BC is parallel to KL — and BK, CL each parallel to CH. ∴ △ EDB = △ BDC and △ ABE = △ EED ∴ parallelogram BEED is ½ of the triangle EAC similarly for the rest.

No. the last part proves the whole proposition

$$d_t^2 x = -m(d_t s)^2 \cdot dx$$
$$= -m\, d_t s \cdot d_t x$$

$$\frac{d_t^2 x}{d_t x} = -m\, d_t s$$

$$\therefore \log d_t x = -ms + C$$

at point of projection $s = 0$. $d_t x = v\cos\alpha$

$$\therefore C = \log v\cos\alpha$$

$$\therefore \log d_t x - \log v\cos\alpha = -ms$$

$$\log \left| \frac{d_t x}{v\cos\alpha} \right| = -ms$$

$$\frac{d_t x}{v\cos\alpha} = E^{-ms}$$

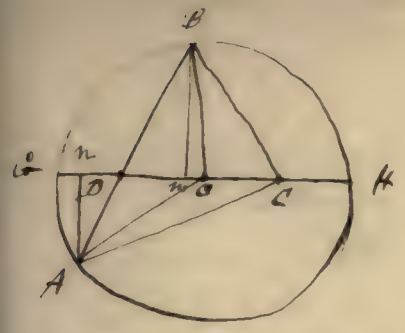
$$d_t x = v\cos\alpha \cdot E^{-ms}$$

$$dy = p \cdot dx$$

$$d^2y = dp \cdot dx + p \cdot d^2x$$

$$d^2y = -g - m\,ds \cdot dy = dp \cdot dx + p\,m\,ds \cdot dx.$$

$$-g = dp \cdot dx.$$



Two points (D. C) are taken in the diameter of a ⊙ at any equal distances from the centre, through one of these (D) draw any chord (A.B.) and join its extremities to the other given point — the Δ thus formed has the sum of its squares invariable

Let O be the centre of the ⊙, and from B & A draw Bm, An ⊥ to the diameter GH.

Then $DB^2 = DO^2 + OB^2 - 2DO \times Om$
$BC^2 = BO^2 + OC^2 + 2CO \times Om$ } But $DO = CO$

$\therefore DB^2 + BC^2 = 2DO^2 + 2OB^2$

and since OB is the radius of the circle $BB^2 + BC^2$ is invariable as long as D & C remain the same.

Similarly it may be shown that $AO^2 + AC^2 = 2AO^2 + 2DO^2$

also $AD \times DB = GD \times DH \therefore 2AD \times DB = 2GD \times DH$

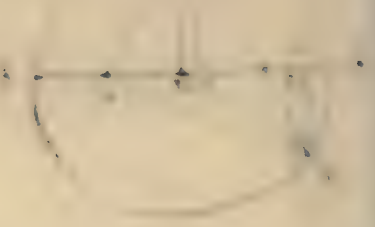
Before collecting all these results together we have

$(AO^2 + 2AO.DB + DB^2) + BC^2 + AC^2 = 4AO^2 + 4DO^2 + 2GD \times DH.$

or $AB^2 + BC^2 + AC^2 = GH^2 + DC^2 + 2GD \times DH$

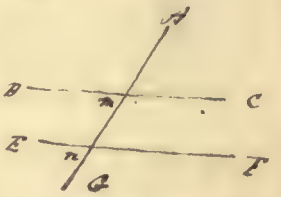
which remains invariable whatever be the position of the angular points A & B.

AA'BB' is a □ having a given angle at A, and also its perimeter of given quantity — Find the locus of P, for all such □'s and construct it

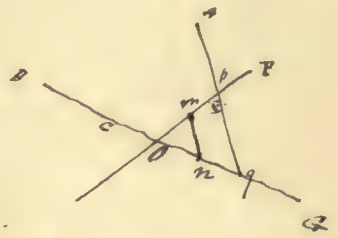


Geometrical Problem.

From a given point to draw a straight line making equal angles with two given straight lines



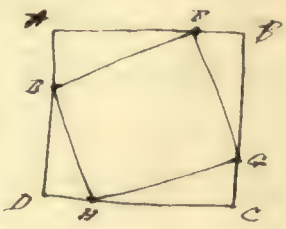
First let the given straight lines be parallel as BC, EF and let A be the given point. Draw a straight line AG cutting BC, EF and it will make the exterior ∠ ANC = interior and opposite upon the same side ANF or AG makes equal angles with BC, EF.



Next let BC and EF be not parallel they shall therefore meet if produced far enough, either towards C F or B E. Let them meet towards CF in O and be continued indefinitely in the same direction. Take any equal distances m, n in OF. OA and join m.n. Through A draw A, q parallel to m.n. and because p.q is drawn parallel to m.n one of the sides of the triangle Amn. Op is to oq as om is to on But om is equal to on, consequently op is equal to oq and the angle opq is equal to the angle oqp, or AqB — Now the angle opq is equal to the opposite vertical angle ApF; therefore the angle AqB, is equal to the angle ApF or from the given point A, a straight line Apq had been drawn making equal angles with the two given straight lines BC, EF which was to be done —

Lemma — If four points be taken in the sides of a square, equally distant from the opposite points, the figure formed by joining them will be a square

Let ABCD be a square, and let the points E, F, G, H be equally distant from A, B, C, D respectively, and join EF, FG, GH, HE, EFGH shall be a square —

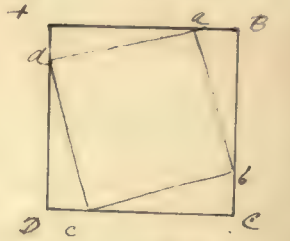


For because AF = BG and FB = AE the two AF, AE are equal to the two BA, EB respectively, and the right angle AA = right angle at B therefore the base EF = the base EG. similarly it may be shown that AH, and EH. are each equal to EF or EG Therefore the four straight lines EF, FG, GH, HE are equal to each other again because the angles AFE BFG, GEB are equal to two right angles, and the angles AEE, AEF, EAF are equal to two right angles being the angles upon A the former are equal to the latter. wherefore the angle AEF is common, and the angle AEF equal to the angle GFB, therefore the remaining angle BAF is equal to the remaining angle BEG — but BAF is by hypoth. a right angle, consequently angle EFG is a right angle — Similarly it may be shown that the angles EGH, GHE, HEF are right angles. Therefore the figure EFGH is both equilateral and it has been shown to be equiangular, it is therefore a square. Q.E.D.

Geometrical Problem.

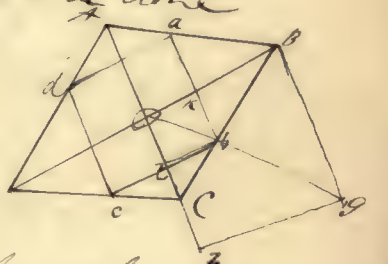
It is required to inscribe a square in a given square so that one of its angles shall be also given, lying in one of the sides of the square; also to inscribe a square in a given rhombus.

Let ABCD be the given square and E the proposed square be required to have one of its angular points in a.



Make aB, BC, CD, and Dd equal ∴ Aa, Bb, Cc, Dd are all equal. Join ab, bc, cd, da. Then in the Δs aBb, bCc, the sides aB, Bb are equal to bC, Cc and the right ∠ aBb = right ∠ bCc ∴ ab = bc similarly ad, dc are each equal to cb, ab ∴ abcd is equilateral. It is also rectangular for the angles Bab, Bba are equal to a right angle and the angle Bba = ∠ Aab ∴ the angles Aab, Bab are equal to a right angle and the angle Bab is a right angle — similarly it may be shown that each of the angles at b, c, d are right angles ∴ abcd is a rectangle, and it has been shown to be equilateral, it is therefore a square, and one of its angular points passes through a which was to be done

Let ABCD be the rhombus and draw the diagonals AC, BD which cut each at right angles ∴ the triangles ADC, CDO, CBO, ABC are equal and similar. Upon BC describe a square and join Og. Then through b draw bk, Bl parallel to Bg, gb. Then the parallelogram kbBl is double the triangle BOC, ABC are equal and similar, hence can be inscribed in each of the same triangle equal to kBlb and consequently bk and kl are in the same straight line, and similarly of the other squares. And each of the angles at a, d, c, b is a right angle lie in one of the angles of the square and the sides ab, bc, cd, da are also equal ∴ abcd is both equilateral and rectangular, and it is inscribed in the rhombus ABCD which was to be done.

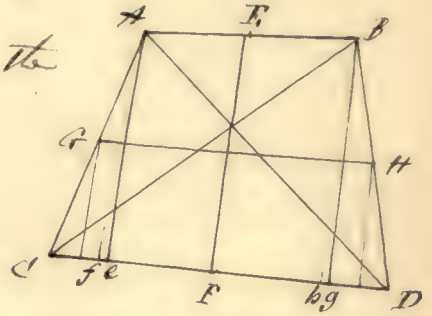


Conical Problem

If two right lines be drawn to join the middle points of the opposite sides of any quadrilateral figure the sum of their squares will be equal to the sum of the squares of the two diagonals.

Let ABCD be a quadrilateral, and E. H. F. G the points of bisection

$$AD^2 + CB^2 = 2(EF^2 + GH^2)$$



draw Ae. Bg parallel to EF. Then if these lines are at right angles to CD we have

$$AD^2 = eD^2 + Ae^2 = GH^2 + EF^2 \text{ for } GH \text{ evidently} = \tfrac{1}{2}(AE + CD)$$
$$= Eg + CF = Cg = \tfrac{1}{2}CD$$

and $CB^2 = Bg^2 + Cg^2 = GH^2 + EF^2$

$\therefore AD^2 + CB^2 = 2\{GH^2 + EF^2\}$ $\quad Cg = CF + Eg = FD + Fe = eD.$

But if Ae. Bg be not at right angles to CD. draw Af. Bh at right angles to CD. Then because $fh = AB = eg$ we have $fe = hg$.

Also if the angle AeD is obtuse, the angle Aec = angle BgC is acute and vice versa.

$$\text{Now } AD^2 = Ae^2 + eD^2 \pm 2fe \cdot eD = EF^2 + GH^2 \pm 2fe \cdot eD$$
$$CB^2 = Bg^2 + gC^2 \mp hg \cdot gC = EF^2 + GH^2 \mp hg \cdot gC$$

using the upper or lower signs according as AeD is obtuse or otherwise

and since fe has been shown $= hg$

and eD ——— $= gC$

in either case by addition we have

$$AD^2 + CB^2 = 2\{EF^2 + GH^2\}$$

which also holds good when Ae is at right angles

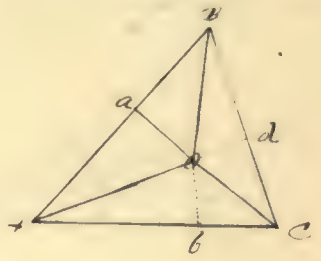
Therefore in all cases we have

$$\tfrac{1}{2}(AD^2 + CB^2) = EF^2 + GH^2$$

Geometrical Problem.

Straight lines which bisect each of the three angles of a triangle will intersect each other in the same point.

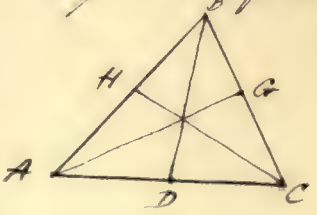
Let $\triangle ABC$ be a triangle and let the straight lines AD, CD be drawn bisecting the angles BAC, BCA, and meeting each other in the point D, and join DB. Let Da, Db, Dd be drawn perpendicular to the sides.



Then $Aa = Ab$, for the triangles have $2a = 2b$ = a right angle and one side common. For the same reason $Bc = Bd$. And the three lines Aa, BD, Dd, are equal by case IV. Then in the triangles BAa, BDd, Aa is equal to Dd, and BD is common the square of the two sides BD, Dd = to the sides BA, Aa, and the right angle BdA = right angle BaA, therefore the angle ABD = angle DBd. or ABC is bisected by DB, which passes through the point of intersection of AD, CD which bisect the angles BAC and BCA. Therefore the three straight lines AD, BD, CD which intersect the three angles pass through the same point —

If right lines be drawn from each of the angles of a triangle to the middle of the opposite sides, three times the sum of the squares of the three sides of the triangle will be equal to four times the sum of the squares of the three bisecting lines

Let FD be drawn bisecting the side AC. And first let BD be at right angles to AC



Then $AB^2 + BC^2 = 2(BD^2 + AD^2)$

Next let BDA be an acute angle

Then $AB^2 = AD^2 + DB^2 + 2AD \times DB$ and $BC^2 = AD^2 + DC^2 - 2DC \times BD$

$\therefore AB^2 + BC^2 = 2(BD^2 + AD^2)$ as before, and similarly if BDA be obtuse

Similarly in all cases $CB^2 + AC^2 = 2(HC^2 + HB^2)$

$$AB^2 + AC^2 = 2(AG^2 + GC^2)$$

$$\therefore 4(AB^2 + BC^2 + AC^2) = 4(BD^2 + HG^2 + AG^2) + 4(AD^2 + AH^2 + BG^2)$$

But $4(AD^2 + AH^2 + BG^2) = AB^2 + BC^2 + AC^2$

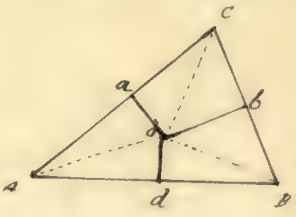
$$\therefore 3(AB^2 + BC^2 + AC^2) = 4(BD^2 + HG^2 + AG^2)$$

which was to be shewn.

Geometrical Problem.

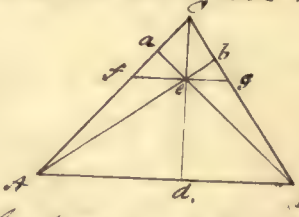
(7) If perpendiculars be drawn through a triangle from the middles of each of its sides, they will intersect each other in the same point.

Let ABC be the triangle whose sides are bisected in the points a, b, d. the perpendiculars from these points shall meet in one point.



Draw the perpendiculars oa, ob, od intersecting in o and join oA, oC, ob, oB. Then by Euclid oA, oB. oC are equal to each other. And because oC = oB, and Cb = Bb, and the base bo is common. the angle Cbo is equal to the angle Bbo. Wherefore bo is at right angles to BC and it passes through o, the point of intersection of the perpendiculars from the points d.a. therefore the perpendiculars from a.b.d pass through the same point.

(8) If perpendiculars be demitted from each of the three angles of a triangle to the opposite sides, they will intersect each other in the same point



Draw the straight lines Ab, Ba, at right angles to CB, Cf, intersecting each other in e, join Ce and produce it to d. and through e draw fg parallel to AB. then the angle Ceb is equal to the angle Aeb

" " " angles Cae, Ace

" " " angles CBa, eBb

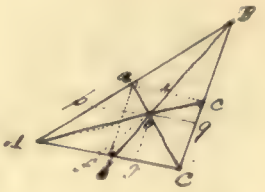
" " " angle CBf.

" " " angle Cge

and the angle Cfg is common to the 2 Δs eCg, Ceb. therefore the remaining angle Cbe is equal to the remaining angle Ceg. consequently Ceg is a right angle; therefore CdB is also a right angle for fg is parallel to AB. wherefore Cd cuts AB at right angles, and it passes through e the point of intersection of Ba, Ab which are the perpendiculars from CA, CB. and also through C. wherefore the perpendiculars demitted from the angular points upon the sides intersect each other in one point.

Geometrical Problem.

If straight lines be drawn from each of the three angles of a triangle to the middle of their opposite sides they will intersect each other in the same point.



Let ABC be the given triangle and a, b, c the points of bisection and join Ac, Bb, Ca. These straight lines shall intersect in one point.

Join ac. and because AB, BC are bisected in a, c. ac is parallel to AC. and since Ab = bc; aa = xc Through x draw fg parallel to AC, then af : Aa :: Cg : CC . But Af : Ag :: ao : ac and cg : CC :: Co : CA :: Ac : xc :: aC : ao. Through a draw Of, Og parallel to Aa, Cc

Then Ac : ac :: Ag : AC and aC : ao

Then AC : oc :: AC : gc and aC : ao :: AC : Af

:: AC : gC :: AC : Af :: Af = gc and fo = og since fg is parallel to Ac. consequently aC and Ac intersect each other at equal distances from AB, BC on a line fg parallel to AC

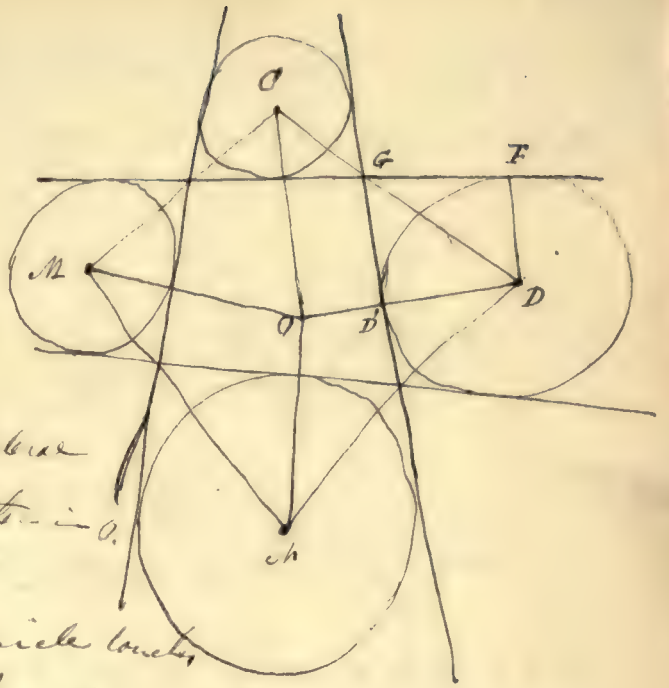
Also because ACca is a parallelogram and ax = xC, and Ab = bc the straight line xb preserves equal distances from Aa, cC. But xb is in the same straight line with Bb; consequently Bb is at equal distances from AB, BC along lines parallel to AC — But the straight lines aC, Ac have been shown to intersect each other at the same distance from AB, BC: Therefore the point of the intersection of the latter will coincide with a point in the former. or the three lines Bb, aC, Ac intersect each other in the same point which was to be shown.

or. Bf : Ba :: Bg : Bc; but Bf : BA :: fo : Ab and Bg : BC :: go : bc

:: fo : Ab :: go : Bc and since Ab = Bc, fo = go. Therefore Bb preserves the same distance from AB, BC.

Four circles are drawn of which each touches one side of a quadrilateral figure and the adjacent sides produced; shew that the centres of the four circles will lie in the circumference of a circle

Let C and O be the centres of the circles, and through the points where the sides of the quadrilateral touch the circles, draw CO, DO, meeting each other in O, then the problem is to shew that OC = OD



Let DE be drawn to the point where the circle touches the side of the quadrilateral produced, then because the angles DD'G, DEG are right angles, $D'D^2 + DG^2 = GE^2 + ED^2$ ∴ DG = GE

and in the triangles D'GD, EGD the two sides DG, GD are equal to the sides GD, GE; and the base DD' = the base DE; consequently the angle DGE = angle DGD. But DGE equals the opposite vertical angle D'GD, CGE ∴ the angle DGD' = angle CGE

again since the angles GCO, CGE = a right angle

and angles DGD', DBG = do

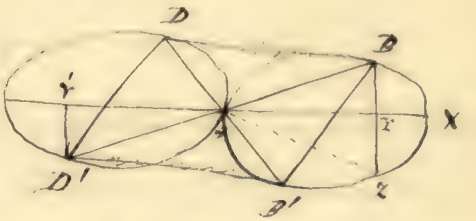
of which the angle DGD = angle CGE

Therefore the remaining angle GCO = remaining angle D'DG.

and the side OC = side OD

Similarly, it may be shewn that OD, OC, OM and ON is equal to OC or OD and therefore the 4 straight lines OD, OC, OM, ON are equal to one another, and the circle described from the centre O at the distance of either of the four will pass through the extremities of the others; and consequently the four points B, C, M, N (which are the centres of the circles touching the sides of the quadrilateral) are all in the circumference of a circle —

Let A be common vertex of two similar ellipses ABB', ADD' whose greater axes coincide from chords ABD, $AB'D'$ and join BB', DD'; BB' will be parallel to DD'



From B draw BY at right angles to AX the greater axis of both the ellipses, and produce BY to Z and join AZ

Then $x=a$ the equation to the ellipse $y = \pm \frac{b}{a}\sqrt{a^2-x^2}$ gives two equal values of y but of contrary signs because $YZ = BY$ and the angle $BAY = \angle BAZ$—

Also because the two ellipses are similar, AY bisects $= AY$ and DY be drawn perpendicular to it the triangles DYA, ZYA are similar and equal $\therefore$ the angle YAD = angle YAZ = $\angle YAB$

$\therefore$ DA, AB are in the same straight line

and conversely if AD and AB are in the same straight line $AD = AB$

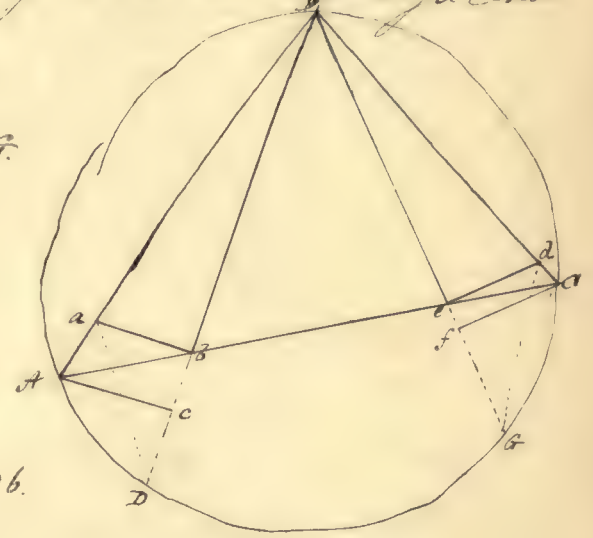
similarly $AD = AB'$ and the angle DAD' = angle BAB' being opposite vertical angles. Therefore the triangles DAD', BAB' have the angle $AD'D$ = angle $DB'B$ and they are alternate angles — Therefore DD' is parallel to BB'

Prop XVII.

Parallel $AB \times CD + AC \times DB = AD \times BC$

$$AB \times CD + AC \times DB = AD \times CD + (AC + CD) DB$$
$$= AD \times CD + AD \times DB$$
$$= AD(CD + DB) = AD \times CB.$$

Prop XVIII. If two right lines be drawn from the vertex of a triangle to the base, so as to make equal angles with the sides, the squares of these two sides will be to each other as the rectangles under the adjacent segments of the base.



Then by Euclid $AB^2 : BC^2 :: AB \times BC :: Ce : eA$.

Draw Ac & ab; Cf & de, perpendicularly to Bd, BG.

Then by Equiangular triangles,

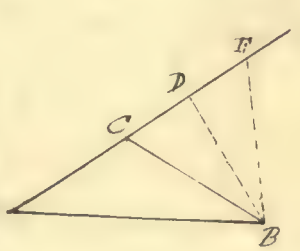
$$BA : AC :: BC : Cf$$
$$BA : BC :: AC : Cf$$
$$ab : de$$
$$BA^2 : BC^2 :: ab^2 : de^2$$
$$:: Bb \times BD :: Be \times eG$$

for ab is a mean prop between Bb and Db.

$$:: AB \times BC : Ce : eA$$

Prop XIX. If the vertical angle of a triangle be two third parts of two right angles, the square of the base will be equal to the sum of the squares of the two sides, together with the rectangle of those sides.

Because the angle ACB contains 120° degrees the adjacent angle BCD contains 60°. Draw BD at right angles to AC produced. Then $\angle CBD = 30^\circ$ make $\angle DBE = \angle CBD$ Then CBE is both isosceles and equiangular, also $CE = 2CD$. $\therefore CB = 2CD$.



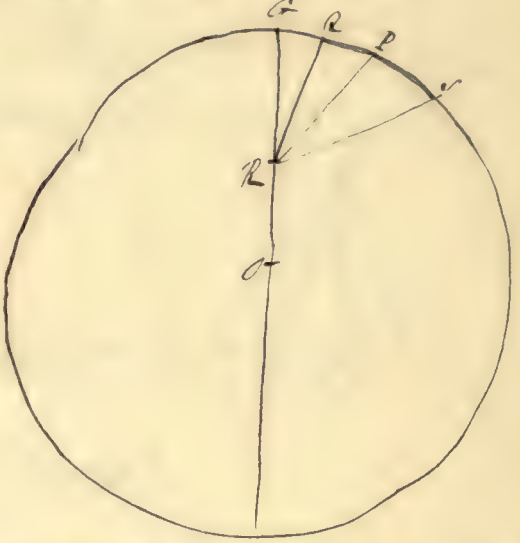
Now by Euclid $AB^2 = AC^2 + CB^2 + 2DC \times AC$

But CB has been shewn to equal $2DC$

$$\therefore AB^2 = AC^2 + CB^2 + BC \times AC$$

Prop. 24. Geometrical Problem.

If from any point in the radius of a circle more than two lines be drawn to the circumference, making equal angles with each of the two nearest the radius will be less than that more remote.



Let O be the centre of the circle
R a point in the radius OC
and let the lines RQ, RP, RS be drawn
making equal angles with each other. Join QP, PS.

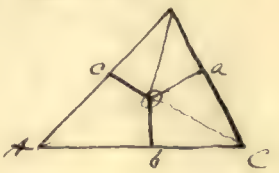
Then since R is not the centre of the
circle by Euc B III. RS > RP
RP > RQ
a fortiori RS > RQ.

and because the triangles RQP, SPR have
PR common and ∠ QPR S = ∠ PRQ, but side RQ < side RS, therefore
base QP is less than base PS. But in equal circles the arcs of
circles are as the chords that subtend them.

∴ arc QP is less than arc PS.

The area of a plane triangle equals the rectangle of half
the sum of the sides and the radius of the inscribed circle.

Let OC = Oa = Ob = r be the radii of the inscribed
circle which are perpendicular to and bisect the
sides adjacent to them.



∴ area Δ AOB = $\frac{1}{2}(AB \times OC) = \frac{1}{2} AB \times r$

" Δ BOC = $\frac{1}{2}(BC \times Oa) = \frac{1}{2} BC \times r$

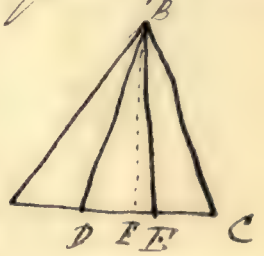
" " Δ AOC = $\frac{1}{2}(AC \times Ob) = \frac{1}{2} AC \times r$

∴ area of the triangle ABC = $\frac{1}{2}(AB + BC \times AC) \times r$

Q. E. D.

Geometrical Problem.

If the base of a triangle be divided into three equal parts, and two lines be drawn from the vertex to the points of division. the sum of the squares of the two sides of the triangle will be equal to the sum of the squares of the two lines before mentioned, together with four times the square of one of the equal parts of the base.



$AB^2 + BE^2 = BE^2 + DB^2 + 4AD^2$

$2BE^2 + AE^2 + EC^2 = 2BE^2 + DE^2 + FE^2 + 4AD^2$

$AD^2 + DE^2 + 2AD.DE$
$+ FE^2 + EC^2 + 2DE.FE \Big\} = DE^2 + FE^2 + 4AD^2$

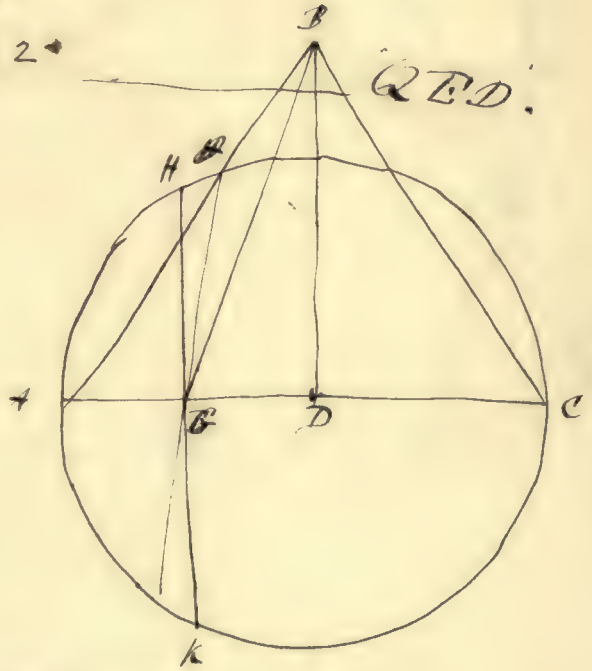
$2(FE + DE)FD = 2AD^2$

$2AD \times DE = 2AD^2$

But $2AD \times DE = 2AD^2$

$\therefore AB^2 + BE^2 = BE^2 + DB^2 + 4AD^2$ Q.E.D.

Prop 22. If ABC is an equilateral Δ



$AC^2 = BG^2 + HG^2$

$AB^2 = AG^2 + GB^2 + 2AG.GD$

$= GB^2 + AG(2GD + AG)$

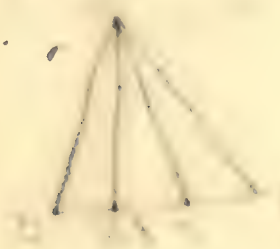
$= GB^2 + AG \times GC$

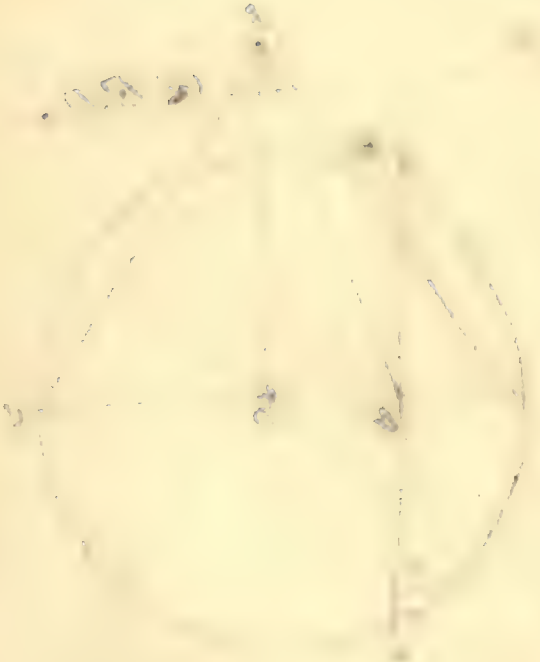
$= GB^2 + HG \times GK$

$= GB^2 + GH^2$

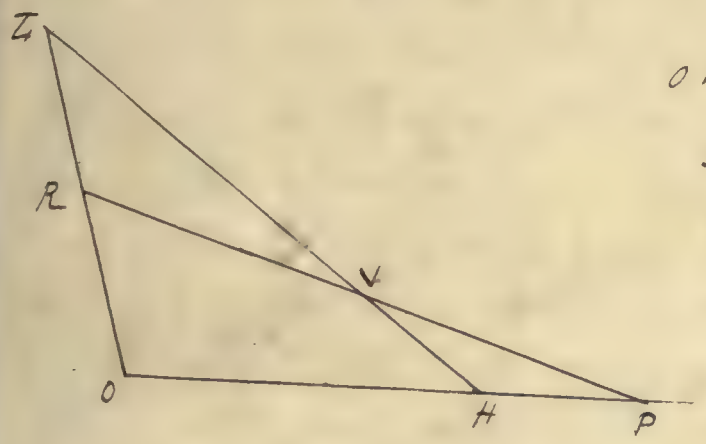
But $AB = AC$

$\therefore AC^2 = GB^2 + GH^2$ — Q.E.D.





KLPQR are 2 Δs so that if the straight lines HP KQ LR meet in one point the intersections of KL & QR LH & RP HK & PQ lie in a straight line



$OH = a \quad OP = a' \quad OL = c \quad OR = c'$.

x & y coordes of V referred to axes OH OL

$$\frac{x}{a} + \frac{y}{c} = 1 \qquad \frac{x}{a'} + \frac{y}{c'} = 1$$

$$\therefore x = \frac{c' - c}{\frac{c'}{a'} - \frac{c}{a}} \qquad y = \frac{a' - a}{\frac{a'}{c'} - \frac{a}{c}}$$

$$\frac{LV}{LH} = \frac{x}{a} \qquad \frac{HV}{LH} = \frac{y}{c} \qquad \frac{RV}{RP} = \frac{x}{a'} \qquad \frac{PV}{RP} = \frac{y}{c'}$$

$$\frac{LV}{HV} = \frac{x}{a} \cdot \frac{c}{y} = \frac{c - c'}{a' - a} \cdot \frac{a'}{c'} \qquad \frac{RV}{PV} = \frac{c - c'}{a' - a} \cdot \frac{a}{c}$$

$OH = a \quad OK = b \quad OL = c$

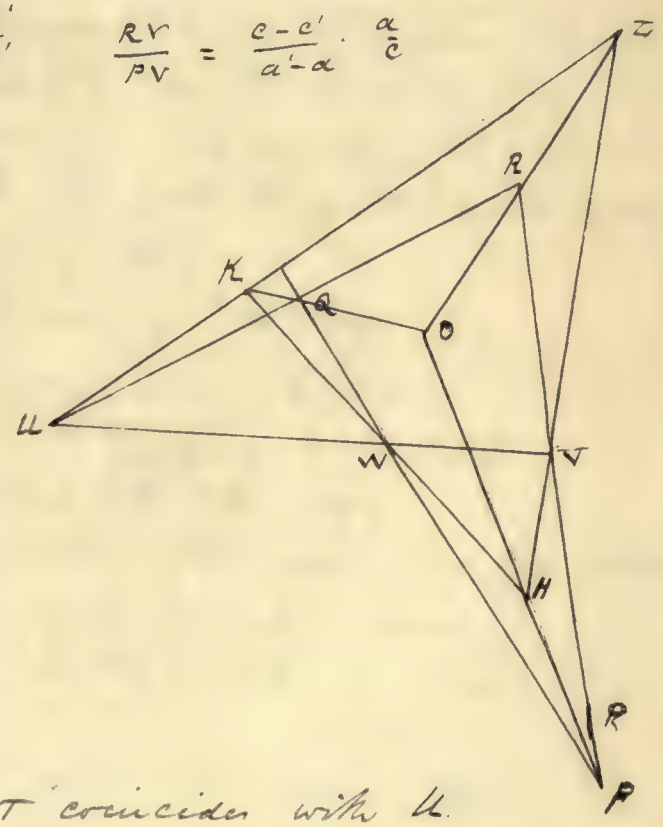
$OP = a' \quad OQ = b' \quad OR = c'$

$$\frac{U}{L} = \frac{b - b'}{c - c'} \cdot \frac{c'}{b'}$$

$$\frac{V}{V} = \frac{a' - a}{c - c'} \cdot \frac{c'}{a'}$$

$$\frac{W}{W} = \frac{a' - a}{b - b'} \cdot \frac{b'}{a'}$$

} UVW are the intersections of LK & QR LH & RP HK & PQ



Let KL VW intersect in T

$$\frac{KT}{LT} = \frac{KW}{LV} \cdot \frac{HV}{HW} = \frac{b - b'}{c - c'} \cdot \frac{c'}{b'}$$

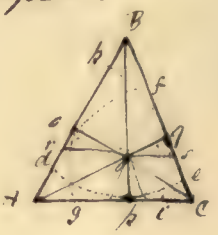
$$\therefore \frac{KT}{LT} = \frac{KU}{LU}$$

$\therefore$ T coincides with U.

Geometrical Problem.

J. Johns. 1841.

The lines joining the angles of a triangle with the points in which the escribed circles touch the opposite sides, meet in a point. Then also that if the base and the sum of the other sides is constant, the locus of the centre of the escribed circle touching the base is an ellipse.



Let ABC be a triangle, fg, de, hk arcs of the escribed circles touching the opposite sides in g, p, o — and let the lines Ag. Bp. Co be drawn from the opposite angles to the points of contact. These lines shall meet in one point.

Because the circles touch the sides of the triangle, and from the points of contact straight lines are drawn to the centre, of the circles which coincide with the angular points A, B C, the lines Ag. Bp. Co and the sides of the triangle do right angles. Let any two of them Ag. Co meet in O. join BO, Op. they shall be in the same straight line and produced to meet AC if they have not that property through O. draw rOs parallel to AC then

$$\angle BOq = \angle AOp = \angle s\ AOB + BAO$$
$$= \angle s\ OCp + OCq$$
$$= \angle pCq = \angle OsB$$

and the angle OBq is common to the triangles OBs OBq

$\therefore$ the $\angle BOs$ = right angle OqB — and rs is parallel to AC $\therefore$ Bp cuts AC at right angles. and since Bp is drawn from the centre perpendicular to AC which touches the circle, it passes through the point of contact.

Also since AB + BC is constant. let it = 2a,

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